

# GENERALIZED GEVREY ULTRADISTRIBUTIONS

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ABSTRACT. We first introduce new algebras of generalized functions containing Gevrey ultradistributions and then develop a Gevrey microlocal analysis suitable for these algebras. Finally, we give an application through an extension of the well-known Hörmander's theorem on the wave front of the product of two distributions.

## 1. INTRODUCTION

The theory of generalized functions initiated by J. F. Colombeau, see [4] and [5], in connection with the problem of multiplication of Schwartz distributions [20], has been developed and applied in nonlinear and linear problems, [5], [17] and [16]. The recent book [7] gives further developments and applications of such generalized functions. Some methods of constructing algebras of generalized functions of Colombeau type are given in [1], [7] and [15].

Ultradistributions, important in theoretical as well applied fields, see [13], [14] and [19], are natural generalization of Schwartz distributions, and the problem of multiplication of ultradistributions is still posed. So, it is natural to search for algebras of generalized functions containing spaces of ultradistributions, to study and to apply them. This is the purpose of this paper.

First, we introduce new differential algebras of generalized Gevrey ultradistributions  $\mathcal{G}^\sigma(\Omega)$  defined on an open set  $\Omega$  of  $\mathbb{R}^n$  as the quotient algebra

$$\mathcal{G}^\sigma(\Omega) = \frac{\mathcal{E}_m^\sigma(\Omega)}{\mathcal{N}^\sigma(\Omega)},$$

where  $\mathcal{E}_m^\sigma(\Omega)$  is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\exists k > 0$ ,  $\exists c > 0$ ,  $\exists \varepsilon_0 \in ]0, 1]$ ,  $\forall \varepsilon \leq \varepsilon_0$ ,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(k\varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

and  $\mathcal{N}^\sigma(\Omega)$  is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\forall k > 0$ ,  $\exists c > 0$ ,  $\exists \varepsilon_0 \in ]0, 1]$ ,  $\forall \varepsilon \leq \varepsilon_0$ ,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

The functor  $\Omega \rightarrow \mathcal{G}^\sigma(\Omega)$  being a sheaf of differential algebras on  $\mathbb{R}^n$ , we show that  $\mathcal{G}^\sigma(\Omega)$  contains the space of Gevrey ultradistributions of order  $(3\sigma - 1)$ , and the following diagram of embeddings is commutative

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$$\begin{array}{ccc} E^\sigma(\Omega) & \rightarrow & \mathcal{G}^\sigma(\Omega) \\ & \searrow & \uparrow \\ & & D'_{3\sigma-1}(\Omega) \end{array}$$

We then develop a Gevrey microlocal analysis adapted to these algebras in the spirit of [10], [19] and [16]. The starting point of the Gevrey microlocal analysis in the framework of the algebra  $\mathcal{G}^\sigma(\Omega)$  consists first in introducing the algebra of regular generalized Gevrey ultradistributions  $\mathcal{G}^{\sigma,\infty}(\Omega)$  and then to prove the following fundamental result

$$\mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega) = E^\sigma(\Omega)$$

The functor  $\Omega \rightarrow \mathcal{G}^{\sigma,\infty}(\Omega)$  is a subsheaf of  $\mathcal{G}^\sigma$ . This permits to define the generalized Gevrey singular support and then, with the help of the Fourier transform, the generalized Gevrey wave front of  $f \in \mathcal{G}^\sigma(\Omega)$ , denoted  $WF_g^\sigma(f)$ , and further to give its main properties, as  $WF_g^\sigma(T) = WF^\sigma(T)$ , if  $T \in D'_{3\sigma-1}(\Omega) \cap \mathcal{G}^\sigma(\Omega)$ , and

$$WF_g^\sigma(P(x, D)f) \subset WF_g^\sigma(f), \forall f \in \mathcal{G}^\sigma(\Omega),$$

if  $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$  is a partial differential operator with  $\mathcal{G}^{\sigma,\infty}(\Omega)$  coefficients.

Let us note that in [3], the authors introduced a general well adapted local and microlocal ultraregular analysis within Colombeau algebra  $\mathcal{G}(\Omega)$ .

Finally, we give an application of the introduced generalized Gevrey microlocal analysis. The product of two generalized Gevrey ultradistributions always exists, but there is no final description of the generalized wave front of this product. Such problem is also still posed in the Colombeau algebra. In [11], the well-known Hörmander's result on the wave front of the product of two distributions, has been extended to the case of two Colombeau generalized functions. We show this result in the case of two generalized Gevrey ultradistributions, namely we obtain the following result : let  $f, g \in \mathcal{G}^\sigma(\Omega)$ , satisfying  $\forall x \in \Omega$ ,

$$(x, 0) \notin WF_g^\sigma(f) + WF_g^\sigma(g),$$

then

$$WF_g^\sigma(fg) \subseteq (WF_g^\sigma(f) + WF_g^\sigma(g)) \cup WF_g^\sigma(f) \cup WF_g^\sigma(g)$$

## 2. GENERALIZED GEVREY ULTRADISTRIBUTIONS

To define the algebra of generalized Gevrey ultradistributions, we first introduce the algebra of moderate elements and its ideal of null elements depending on the Gevrey order  $\sigma \geq 1$ . The set  $\Omega$  is a non void open of  $\mathbb{R}^n$ .

**Definition 1.** *The space of moderate elements, denoted  $\mathcal{E}_m^\sigma(\Omega)$ , is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega$ ,  $\forall \alpha \in \mathbb{Z}_+^m, \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,*

$$(1) \quad \sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp \left( k \varepsilon^{-\frac{1}{2\sigma-1}} \right)$$

*The space of null elements, denoted  $\mathcal{N}^\sigma(\Omega)$ , is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega, \forall \alpha \in \mathbb{Z}_+^m, \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,*

$$(2) \quad \sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp \left( -k \varepsilon^{-\frac{1}{2\sigma-1}} \right)$$

The main properties of the spaces  $\mathcal{E}_m^\sigma(\Omega)$  and  $\mathcal{N}^\sigma(\Omega)$  are given in the following proposition.

**Proposition 1.** 1) The space of moderate elements  $\mathcal{E}_m^\sigma(\Omega)$  is an algebra stable by derivation.

2) The space  $\mathcal{N}^\sigma(\Omega)$  is an ideal of  $\mathcal{E}_m^\sigma(\Omega)$ .

*Proof.* 1) Let  $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$  and  $K$  be a compact of  $\Omega$ , then  $\forall \beta \in \mathbb{Z}_+^m, \exists k_1 = k_1(\beta) > 0, \exists c_1 = c_1(\beta) > 0, \exists \varepsilon_{1\beta} \in ]0, 1], \forall \varepsilon \leq \varepsilon_{1\beta}$ ,

$$(3) \quad \sup_{x \in K} |\partial^\beta f_\varepsilon(x)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

$\forall \beta \in \mathbb{Z}_+^m, \exists k_2 = k_2(\beta) > 0, \exists c_2 = c_2(\beta) > 0, \exists \varepsilon_{2\beta} \in ]0, 1], \forall \varepsilon \leq \varepsilon_{2\beta}$ ,

$$(4) \quad \sup_{x \in K} |\partial^\beta g_\varepsilon(x)| \leq c_2 \exp\left(k_2 \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Let  $\alpha \in \mathbb{Z}_+^m$ , then

$$|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} f_\varepsilon(x)| |\partial^\beta g_\varepsilon(x)|$$

For  $k = \max\{k_1(\beta) : \beta \leq \alpha\} + \max\{k_2(\beta) : \beta \leq \alpha\}$ ,  $\varepsilon \leq \min\{\varepsilon_{1\beta}, \varepsilon_{2\beta} : |\beta| \leq |\alpha|\}$  and  $x \in K$ , we have

$$\begin{aligned} \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \exp\left(-k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) |\partial^{\alpha-\beta} f_\varepsilon(x)| \\ &\quad \times \exp\left(-k_2 \varepsilon^{-\frac{1}{2\sigma-1}}\right) |\partial^\beta g_\varepsilon(x)| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1(\alpha - \beta) c_2(\beta) = c(\alpha), \end{aligned}$$

i.e.  $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$ .

It is clear, from (3) that for every compact  $K$  of  $\Omega$ ,  $\forall \beta \in \mathbb{Z}_+^m, \exists k_1 = k_1(\beta + 1) > 0, \exists c_1 = c_1(\beta + 1) > 0, \exists \varepsilon_{1\beta} \in ]0, 1]$  such that  $\forall x \in K, \forall \varepsilon \leq \varepsilon_{1\beta}$ ,

$$|\partial^\beta (\partial f_\varepsilon)(x)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

i.e.  $(\partial f_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$ .

2) If  $(g_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ , for every  $K$  compact of  $\Omega$ ,  $\forall \beta \in \mathbb{Z}_+^m, \forall k_2 > 0, \exists c_2 = c_2(\beta, k_2) > 0, \exists \varepsilon_{2\beta} \in ]0, 1]$ ,

$$|\partial^\alpha g_\varepsilon(x)| \leq c_2 \exp\left(-k_2 \varepsilon^{-\frac{1}{2\sigma-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$$

Let  $\alpha \in \mathbb{Z}_+^m$  and  $k > 0$ , then

$$\begin{aligned} \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}}\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}}\right) \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} f_\varepsilon(x)| \times \\ &\quad \times |\partial^\beta g_\varepsilon(x)| \end{aligned}$$

Let  $k_2 = \max \{k_1(\beta); \beta \leq \alpha\} + k$  and  $\varepsilon \leq \min \{\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha\}$ , then  $\forall x \in K$ ,

$$\begin{aligned} \exp \left( k\varepsilon^{-\frac{1}{2\sigma-1}} \right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left[ \exp \left( -k_1 \varepsilon^{-\frac{1}{2\sigma-1}} \right) |\partial^{\alpha-\beta} f_\varepsilon(x)| \right. \\ &\quad \times \exp \left( k_2 \varepsilon^{-\frac{1}{2\sigma-1}} \right) |\partial^\beta g_\varepsilon(x)| \Big] \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1(\alpha - \beta) c_2(\beta, k_2) = c(\alpha, k), \end{aligned}$$

which shows that  $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$  □

**Remark 1.** The algebra of moderate elements  $\mathcal{E}_m^\sigma(\Omega)$  is not necessary stable by  $\sigma$ -ultradifferentiable operators, because the constant  $c$  in (1) depends of  $\alpha$ .

According to the topological construction of Colombeau type algebras of generalized functions, we introduce the desired algebras.

**Definition 2.** The algebra of generalized Gevrey ultradistributions of order  $\sigma \geq 1$ , denoted  $\mathcal{G}^\sigma(\Omega)$ , is the quotient algebra

$$\mathcal{G}^\sigma(\Omega) = \frac{\mathcal{E}_m^\sigma(\Omega)}{\mathcal{N}^\sigma(\Omega)}$$

A comparison of the structure of our algebras  $\mathcal{G}^\sigma(\Omega)$  and the Colombeau algebra  $\mathcal{G}(\Omega)$  is given in the following remark.

**Remark 2.** The Colombeau algebra  $\mathcal{G}(\Omega) := \frac{\mathcal{E}_m(\Omega)}{\mathcal{N}(\Omega)}$ , where  $\mathcal{E}_m(\Omega)$  is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega$ ,  $\forall \alpha \in \mathbb{Z}_+^m, \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c\varepsilon^{-k},$$

and  $\mathcal{N}(\Omega)$  is the space of  $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$  satisfying for every compact  $K$  of  $\Omega, \forall \alpha \in \mathbb{Z}_+^m, \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c\varepsilon^k$$

Due to the inequality

$$\exp \left( -\varepsilon^{-\frac{1}{2\sigma-1}} \right) \leq \varepsilon, \forall \varepsilon \in ]0, 1],$$

we have the strict inclusions  $\mathcal{N}^\sigma(\Omega) \subset \mathcal{N}^\tau(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}_m(\Omega) \subset \mathcal{E}_m^\tau(\Omega) \subset \mathcal{E}_m^\sigma(\Omega)$ , with  $\sigma < \tau$ .

We have the null characterization of the ideal  $\mathcal{N}^\sigma(\Omega)$ .

**Proposition 2.** Let  $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$ , then  $(u_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$  if and only if for every compact  $K$  of  $\Omega$ ,  $\forall k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,

$$(5) \quad \sup_{x \in K} |f_\varepsilon(x)| \leq c \exp \left( -k\varepsilon^{-\frac{1}{2\sigma-1}} \right)$$

*Proof.* Let  $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$  satisfying (5), we will show that  $(\partial_i u_\varepsilon)_\varepsilon$  satisfy (5) when  $i = 1, \dots, m$ , and then it will follow by induction that  $(u_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ .

Suppose that  $u_\varepsilon$  has a real values, in the complex case we do the calculus separately for the real and imaginary part of  $u_\varepsilon$ . Let  $K$  be a compact of  $\Omega$ , for  $\delta = \min(1, \text{dist}(K, \partial\Omega))$ , set

$L = K + \overline{B\left(0, \frac{\delta}{2}\right)}$ , then  $K \subset\subset L \subset\subset \Omega$ . By the moderateness of  $(u_\epsilon)_\epsilon$ , we have  $\exists k_1 > 0, \exists c_1 > 0, \exists \varepsilon_1 \in ]0, 1], \forall \varepsilon \leq \varepsilon_1$

$$(6) \quad \sup_{x \in L} |\partial_i^2 u_\epsilon(x)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

By the assumption (5),  $\forall k > 0, \exists c_2 > 0, \exists \varepsilon_2 \in ]0, 1], \forall \varepsilon \leq \varepsilon_2$

$$(7) \quad \sup_{x \in L} |u_\epsilon(x)| \leq c_2 \exp\left(-(2k + k_1) \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Let  $x \in K, \varepsilon$  sufficiently small and  $r = \exp\left(-(k + k_1) \varepsilon^{-\frac{1}{2\sigma-1}}\right) < \frac{\delta}{2}$ . By Taylor's formula, we have

$$\partial_i u_\epsilon(x) = \frac{(u_\epsilon(x + r e_i) - u_\epsilon(x))}{r} - \frac{1}{2} \partial_i^2 u_\epsilon(x + \theta r e_i) r,$$

where  $e_i$  is  $i^{th}$  vector of the canonical base of  $\mathbb{R}^m$ , hence  $(x + \theta r e_i) \in L$ , and then

$$|\partial_i u_\epsilon(x)| \leq |u_\epsilon(x + r e_i) - u_\epsilon(x)| r^{-1} + \frac{1}{2} |\partial_i^2 u_\epsilon(x + \theta r e_i)| r$$

From (6) and (7) :  $|u_\epsilon(x + r e_i) - u_\epsilon(x)| r^{-1} \leq c_2 \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$  and  $|\partial_i^2 u_\epsilon(x + \theta r e_i)| r \leq c_1 \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$ , so

$$|\partial_i u_\epsilon(x)| \leq c \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

which gives the proof.  $\square$

**Proposition 3.** *If  $P$  is a polynomial function and  $f = cl(f_\epsilon)_\epsilon \in \mathcal{G}^\sigma(\Omega)$ , then  $P(f) = (P(f_\epsilon))_\epsilon + \mathcal{N}^\sigma(\Omega)$  is well defined element of  $\mathcal{G}^\sigma(\Omega)$ .*

*Proof.* Let  $(f_\epsilon)_\epsilon \in \mathcal{E}_m^\sigma(\Omega)$ ,  $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$  and  $K$  be a compact of  $\Omega$ , then we have  $\forall \alpha \in \mathbb{Z}_+^m, \exists k = k(\alpha) > 0, \exists c = c(\alpha) > 0, \exists \varepsilon_0 = \varepsilon(\alpha) \in ]0, 1], \forall \varepsilon \leq \varepsilon_0$ ,

$$(8) \quad \sup_{x \in K} |\partial^\alpha f_\epsilon(x)| \leq c \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Let  $\beta \in \mathbb{Z}_+^m$ , so

$$|\partial^\beta P(f_\epsilon)(x)| \leq \sum_{|\alpha| \leq m} |a_\alpha| |\partial^\beta f_\epsilon^\alpha(x)|,$$

by Leibniz formula and (8), we obtain

$$|\partial^\beta P(f_\epsilon)(x)| \leq \sum_{\substack{|\alpha| \leq m \\ \gamma \leq \beta}} c_{\alpha, \gamma} \left( \exp\left(k_{\alpha, \gamma} \varepsilon^{-\frac{1}{2\sigma-1}}\right) \right)^{n_{\alpha, \gamma}},$$

where  $c_{\alpha, \gamma} > 0$  and  $n_{\alpha, \gamma} \in \mathbb{Z}_+$ . Hence

$$|\partial^\beta P(f_\epsilon)(x)| \leq c \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

One can easily check that if  $(f_\epsilon)_\epsilon \in \mathcal{N}^\sigma(\Omega)$ , then  $(P(f_\epsilon))_\epsilon \in \mathcal{N}^\sigma(\Omega)$   $\square$

The space of functions slowly increasing, denoted  $\mathcal{O}_M(\mathbb{K}^m)$ , is the space of  $C^\infty$ -functions all derivatives growing at most like some power of  $|x|$ , as  $|x| \rightarrow +\infty$ , where  $\mathbb{K}^m \simeq \mathbb{R}^m$  or  $\mathbb{R}^{2m}$ .

**Corollary 4.** *If  $v \in \mathcal{O}_M(\mathbb{K}^m)$  and  $f = (f_1, f_2, \dots, f_m) \in \mathcal{G}^\sigma(\Omega)^m$ , then  $v \circ f := (v \circ f_\varepsilon)_\varepsilon + \mathcal{N}^\sigma(\Omega)$  is a well defined element of  $\mathcal{G}^\sigma(\Omega)$ .*

### 3. GENERALIZED POINT VALUES

The ring of Gevrey generalized complex numbers, denoted  $\mathcal{C}^\sigma$ , is defined by the quotient

$$\mathcal{C}^\sigma = \frac{\mathcal{E}_0^\sigma}{\mathcal{N}_0^\sigma},$$

where

$$\mathcal{E}_0^\sigma = \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{C}^{[0,1]}; \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \text{ such that } \forall \varepsilon \leq \varepsilon_0, |a_\varepsilon| \leq c \exp\left(k\varepsilon^{-\frac{1}{2\sigma-1}}\right) \right\}$$

and

$$\mathcal{N}_0^\sigma = \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{C}^{[0,1]}; \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \text{ such that } \forall \varepsilon \leq \varepsilon_0, |a_\varepsilon| \leq c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right) \right\}$$

It is not difficult to see that  $\mathcal{E}_0^\sigma$  is an algebra and  $\mathcal{N}_0^\sigma$  is an ideal of  $\mathcal{E}_0^\sigma$ . The ring  $\mathcal{C}^\sigma$  motivates the following, easy to prove, result.

**Proposition 5.** *If  $u \in \mathcal{G}^\sigma(\Omega)$  and  $x \in \Omega$ , then the element  $u(x)$  represented by  $(u_\varepsilon(x))_\varepsilon$  is an element of  $\mathcal{C}^\sigma$  independent of the representative  $(u_\varepsilon)_\varepsilon$  of  $u$ .*

A generalized Gevrey ultradistribution is not defined by their point values, we give here an example of generalized Gevrey ultradistribution  $f = [(f_\varepsilon)_\varepsilon] \notin \mathcal{N}^\sigma(\mathbb{R})$ , but  $[(f_\varepsilon(x))_\varepsilon] \in \mathcal{N}_0^\sigma$  for every  $x \in \mathbb{R}$ . Let  $\varphi \in D(\mathbb{R})$  such that  $\varphi(0) \neq 0$ . For  $\varepsilon \in ]0, 1]$ , define

$$f_\varepsilon(x) = x \exp\left(-\varepsilon^{-\frac{1}{2\sigma-1}}\right) \varphi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}$$

It is clear that  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\mathbb{R})$ . Let  $K$  be a compact neighborhood of 0, then

$$\sup_K |f'(x)| \geq |f'_\varepsilon(0)| = \exp\left(-\varepsilon^{-\frac{1}{2\sigma-1}}\right) |\varphi(0)|,$$

which show that  $(f_\varepsilon)_\varepsilon \notin \mathcal{N}^\sigma(\mathbb{R})$ . For any  $x_0 \in \mathbb{R}$ , there exists  $\varepsilon_0$  such that  $\varphi\left(\frac{x_0}{\varepsilon}\right) = 0, \forall \varepsilon \leq \varepsilon_0$ , i.e.  $f(x_0) \in \mathcal{N}_0^\sigma$ .

In order to give a solution to this situation, set

$$(9) \quad \Omega_M^\sigma = \left\{ (x_\varepsilon)_\varepsilon \in \Omega^{[0,1]} : \exists k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, |x_\varepsilon| \leq ce^{k\varepsilon^{-\frac{1}{2\sigma-1}}} \right\}$$

Define in  $\Omega_M^\sigma$  the equivalence relation by

$$(10) \quad x_\varepsilon \sim y_\varepsilon \iff \forall k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, |x_\varepsilon - y_\varepsilon| \leq ce^{-k\varepsilon^{-\frac{1}{2\sigma-1}}}$$

**Definition 3.** *The set  $\tilde{\Omega}^\sigma = \Omega_M^\sigma / \sim$  is called the set of generalized Gevrey points. The set of compactly supported Gevrey points is defined by*

$$(11) \quad \tilde{\Omega}_c^\sigma = \left\{ \tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}^\sigma : \exists K \text{ a compact set of } \Omega, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, x_\varepsilon \in K \right\}$$

**Remark 3.** *It is easy to see that  $\tilde{\Omega}_c^\sigma$ -property does not depend on the choice of the representative.*

**Proposition 6.** Let  $f \in \mathcal{G}^\sigma(\Omega)$  and  $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c^\sigma$ , then the generalized Gevrey point value of  $f$  at  $\tilde{x}$ , i.e.

$$f(\tilde{x}) = [(f_\varepsilon(x_\varepsilon))_\varepsilon]$$

is a well-defined element of the algebra of generalized Gevrey complex numbers  $\mathcal{C}^\sigma$ .

*Proof.* Let  $f \in \mathcal{G}^\sigma(\Omega)$  and  $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c^\sigma$ , there exists a compact  $K$  of  $\Omega$  such that  $x_\varepsilon \in K$  for  $\varepsilon$  small, then  $\exists k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0$ ,

$$|f_\varepsilon(x_\varepsilon)| \leq \sup_{x \in K} |f_\varepsilon(x)| \leq c \exp\left(k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Therefore  $(f_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{E}_0^\sigma$ , and it is clear that if  $f \in \mathcal{N}^\sigma(\Omega)$ , then  $(f_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{N}_0^\sigma$ , i.e.  $f(\tilde{x})$  does not depend on the choice of the representative  $(f_\varepsilon)_\varepsilon$ .

Let now  $\tilde{x} = [(x_\varepsilon)_\varepsilon] \sim \tilde{y} = [(y_\varepsilon)_\varepsilon]$ , then  $\forall k > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0$ ,

$$|x_\varepsilon - y_\varepsilon| \leq c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Since  $(f_\varepsilon)_\varepsilon \in \mathcal{E}^\sigma(\Omega)$ , so for every compact  $K$  of  $\Omega$ ,  $\forall j \in \{1, m\}$ ,  $\exists k_j > 0, \exists c_j > 0, \exists \varepsilon_j > 0, \forall \varepsilon \leq \varepsilon_j$ ,

$$\sup_{x \in K} \left| \frac{\partial}{\partial x_j} f_\varepsilon(x) \right| \leq c_j \exp\left(k_j \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

We have

$$|f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon)| \leq |x_\varepsilon - y_\varepsilon| \sum_{j=1}^m \int_0^1 \left| \left( \frac{\partial}{\partial x_j} f_\varepsilon \right) (x_\varepsilon + t(y_\varepsilon - x_\varepsilon)) \right| dt,$$

and  $x_\varepsilon + t(y_\varepsilon - x_\varepsilon)$  remains within some compact  $K$  of  $\Omega$  for  $\varepsilon \leq \varepsilon'$ . Let  $k' > 0$ , then for  $k + k' = \sup_j k_j$  and  $\varepsilon \leq \min(\varepsilon', \varepsilon_0, \varepsilon_j : j = 1, m)$ , we have

$$|f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon)| \leq c \exp\left(-k' \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

which gives  $(f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon))_\varepsilon \in \mathcal{N}_0^\sigma$ . □

The characterization of nullity of  $f \in \mathcal{G}^\sigma(\Omega)$  is given by the following theorem.

**Theorem 7.** Let  $f \in \mathcal{G}^\sigma(\Omega)$ , then

$$f = 0 \text{ in } \mathcal{G}^\sigma(\Omega) \iff f(\tilde{x}) = 0 \text{ in } \mathcal{C}^\sigma \text{ for all } \tilde{x} \in \tilde{\Omega}_c^\sigma$$

*Proof.* It is easy to see that if  $f \in \mathcal{N}^\sigma(\Omega)$  then  $f(\tilde{x}) \in \mathcal{N}_0^\sigma, \forall \tilde{x} \in \tilde{\Omega}_c^\sigma$ . Suppose that  $f \neq 0$  in  $\mathcal{G}^\sigma(\Omega)$ , then by the characterization of  $\mathcal{N}^\sigma(\Omega)$  we have, there exists a compact  $K$  of  $\Omega$ ,  $\exists k > 0, \forall c > 0, \forall \varepsilon_0 > 0, \exists \varepsilon \leq \varepsilon_0$ ,

$$\sup_K |f_\varepsilon(x)| > c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

So there exists a sequence  $\varepsilon_m \searrow 0$  and  $x_m \in K$  such that  $\forall m \in \mathbb{Z}^+$ ,

$$(12) \quad |f_{\varepsilon_m}(x_m)| > \exp\left(-k\varepsilon_m^{-\frac{1}{2\sigma-1}}\right)$$

For  $\varepsilon > 0$  we set  $x_\varepsilon = x_m$  when  $\varepsilon_{m+1} < \varepsilon \leq \varepsilon_m$ . We have  $(x_\varepsilon)_\varepsilon \in \Omega_M^\sigma$  with values in  $K$ , so  $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c^\sigma$  and (12) means that  $(f_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathcal{N}_0^\sigma$ , i.e.  $f(\tilde{x}) \neq 0$  in  $\mathcal{C}^\sigma$ . □

## 4. EMBEDDING OF GEVREY ULTRADISTRIBUTIONS WITH COMPACT SUPPORT

We recall some definitions and results on Gevrey ultradistributions, see [13], [14] or [19].

**Definition 4.** A function  $f \in E^\sigma(\Omega)$ , if  $f \in C^\infty(\Omega)$  and for every compact  $K$  of  $\Omega$ ,  $\exists c > 0$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} (\alpha!)^\sigma$$

Obviously we have  $E^t(\Omega) \subset E^\sigma(\Omega)$  if  $1 \leq t \leq \sigma$ . It is well known that  $E^1(\Omega) = A(\Omega)$  is the space of all real analytic functions in  $\Omega$ . Denote by  $D^\sigma(\Omega)$  the space  $E^\sigma(\Omega) \cap C_0^\infty(\Omega)$ , then  $D^\sigma(\Omega)$  is non trivial if and only if  $\sigma > 1$ . The topological dual of  $D^\sigma(\Omega)$ , denoted  $D'_\sigma(\Omega)$ , is called the space of Gevrey ultradistributions of order  $\sigma$ . The space  $E'_\sigma(\Omega)$  is the topological dual of  $E^\sigma(\Omega)$  and is identified with the space of Gevrey ultradistributions with compact support.

**Definition 5.** A differential operator of infinite order  $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$  is called a  $\sigma$ -ultradifferential operator, if for every  $h > 0$  there exist  $c > 0$  such that  $\forall \gamma \in \mathbb{Z}_+^m$ ,

$$(13) \quad |a_\gamma| \leq c \frac{h^{|\gamma|}}{(\gamma!)^\sigma}$$

The importance of  $\sigma$ -ultradifferential operators lies in the following result.

**Proposition 8.** Let  $T \in E'_\sigma(\Omega)$ ,  $\sigma > 1$  and  $\text{supp} T \subset K$ , then there exist a  $\sigma$ -ultradifferential operator  $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$ ,  $M > 0$  and continuous functions  $f_\gamma \in C_0(K)$  such that

$$\sup_{\gamma \in \mathbb{Z}_+^m, x \in K} |f_\gamma(x)| \leq M \text{ and}$$

$$T = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma$$

The space  $\mathcal{S}^{(\sigma)}(\mathbb{R}^m)$ ,  $\sigma > 1$ , see [8], is the space of functions  $\varphi \in C^\infty(\mathbb{R}^m)$  such that  $\forall b > 0$ , we have

$$(14) \quad \|\varphi\|_{b,\sigma} = \sup_{\alpha, \beta \in \mathbb{Z}_+^m} \int \frac{|x|^{|\beta|}}{b^{|\alpha+\beta|} \alpha!^\sigma \beta!^\sigma} |\partial^\alpha \varphi(x)| dx < \infty$$

**Lemma 9.** There exists  $\phi \in \mathcal{S}^{(\sigma)}(\mathbb{R}^m)$  satisfying

$$\int \phi(x) dx = 1 \text{ and } \int x^\alpha \phi(x) dx = 0, \forall \alpha \in \mathbb{Z}_+^m \setminus \{0\}$$

*Proof.* For an example of function  $\phi \in \mathcal{S}^{(\sigma)}$  satisfying these conditions, take the Fourier transform of a function of the class  $D^{(\sigma)}(\mathbb{R}^m)$  equal 1 in a neighborhood of the origin. Here  $D^{(\sigma)}(\mathbb{R}^m)$  denotes the projective Gevrey space of order  $\sigma$ , i.e.  $D^{(\sigma)}(\mathbb{R}^m) = E^{(\sigma)}(\mathbb{R}^m) \cap C_0^\infty(\mathbb{R}^m)$ , where  $f \in E^{(\sigma)}(\mathbb{R}^m)$ , if  $f \in C^\infty(\mathbb{R}^m)$  and for every compact subset  $K$  of  $\mathbb{R}^m$ ,  $\forall b > 0$ ,  $\exists c > 0$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,

$$(15) \quad \sup_{x \in K} |\partial^\alpha f(x)| \leq cb^{|\alpha|} (\alpha!)^\sigma$$

□

**Definition 6.** The net  $\phi_\varepsilon = \varepsilon^{-m} \phi(\cdot/\varepsilon)$ ,  $\varepsilon \in ]0, 1]$ , where  $\phi$  satisfies the conditions of lemma 9, is called a net of mollifiers.



The space  $E^\sigma(\Omega)$  is embedded into  $\mathcal{G}^\sigma(\Omega)$  by the standard canonical injection

$$(16) \quad \begin{aligned} I : E^\sigma(\Omega) &\rightarrow \mathcal{G}^\sigma(\Omega) \\ f &\mapsto [f] = cl(f_\varepsilon), \end{aligned}$$

where  $f_\varepsilon = f$ ,  $\forall \varepsilon \in ]0, 1]$ .

The following proposition gives the natural embedding of Gevrey ultradistributions into  $\mathcal{G}^\sigma(\Omega)$ .

**Theorem 10.** *The map*

$$(17) \quad \begin{aligned} J_0 : E'_{3\sigma-1}(\Omega) &\rightarrow \mathcal{G}^\sigma(\Omega) \\ T &\mapsto [T] = cl\left((T * \phi_\varepsilon)_{/\Omega}\right)_\varepsilon \end{aligned}$$

*is an embedding.*

*Proof.* Let  $T \in E'_{3\sigma-1}(\Omega)$  with  $\text{supp} T \subset K$ , then there exists an  $(3\sigma-1)$ -ultradifferential operator  $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$  and continuous functions  $f_\gamma$  with  $\text{supp} f_\gamma \subset K$ ,  $\forall \gamma \in \mathbb{Z}_+^m$ , and

$\sup_{\gamma \in \mathbb{Z}_+^m, x \in K} |f_\gamma(x)| \leq M$ , such that

$$T = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma$$

We have

$$T * \phi_\varepsilon(x) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma \frac{(-1)^{|\gamma|}}{\varepsilon^{|\gamma|}} \int f_\gamma(x + \varepsilon y) D^\gamma \phi(y) dy$$

Let  $\alpha \in \mathbb{Z}_+^m$ , then

$$|\partial^\alpha(T * \phi_\varepsilon(x))| \leq \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| |D^{\gamma+\alpha} \phi(y)| dy$$

From (13) and the inequality

$$(18) \quad (\beta + \alpha)!^t \leq 2^{t|\beta+\alpha|} \alpha!^t \beta!^t, \quad \forall t \geq 1,$$

we have,  $\forall h > 0, \exists c > 0$ , such that

$$\begin{aligned} |\partial^\alpha(T * \phi_\varepsilon(x))| &\leq \sum_{\gamma \in \mathbb{Z}_+^m} c \frac{h^{|\gamma|}}{\gamma!^{3\sigma-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| |D^{\gamma+\alpha} \phi(y)| dy \\ &\leq \sum_{\gamma \in \mathbb{Z}_+^m} c \alpha!^{3\sigma-1} \frac{2^{(3\sigma-1)|\gamma+\alpha|} h^{|\gamma|}}{(\gamma + \alpha)!^{2\sigma-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} b^{|\gamma+\alpha|} \times \\ &\quad \times \int |f_\gamma(x + \varepsilon y)| \frac{|D^{\gamma+\alpha} \phi(y)|}{b^{|\gamma+\alpha|} (\gamma + \alpha)!^\sigma} dy, \end{aligned}$$

then for  $h > \frac{1}{2}$ ,

$$\begin{aligned} \frac{1}{\alpha!^{3\sigma-1}} |\partial^\alpha(T * \phi_\varepsilon(x))| &\leq \|\phi\|_{b,\sigma} Mc \sum_{\gamma \in \mathbb{Z}_+^m} 2^{-|\gamma|} \frac{(2^{3\sigma} b h)^{|\gamma+\alpha|}}{(\gamma + \alpha)!^{2\sigma-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right), \end{aligned}$$

i.e.

$$(19) \quad |\partial^\alpha (T * \phi_\varepsilon(x))| \leq c(\alpha) \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

where  $k_1 = (2\sigma - 1) (2^{3\sigma} b h)^{\frac{1}{2\sigma-1}}$ .

Suppose that  $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ , then for every compact  $L$  of  $\Omega$ ,  $\exists c > 0, \forall k > 0, \exists \varepsilon_0 \in ]0, 1]$ ,

$$(20) \quad |T * \phi_\varepsilon(x)| \leq c \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right), \forall x \in L, \forall \varepsilon \leq \varepsilon_0$$

Let  $\chi \in D^{3\sigma-1}(\Omega)$  and  $\chi = 1$  in a neighborhood of  $K$ , then  $\forall \psi \in E^{3\sigma-1}(\Omega)$ ,

$$\langle T, \psi \rangle = \langle T, \chi \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int (T * \phi_\varepsilon)(x) \chi(x) \psi(x) dx$$

Consequently, from (20), we obtain

$$\left| \int (T * \phi_\varepsilon)(x) \chi(x) \psi(x) dx \right| \leq c \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right), \forall \varepsilon \leq \varepsilon_0,$$

which gives  $\langle T, \psi \rangle = 0$

□

**Remark 4.** We have  $c(\alpha) = \alpha!^{3\sigma-1} \|\phi\|_{b,\sigma} M c$  in (19).

In order to show the commutativity of the following diagram of embeddings

$$\begin{array}{ccc} D^\sigma(\Omega) & \rightarrow & \mathcal{G}^\sigma(\Omega) \\ & \searrow & \uparrow \\ & & E'_{3\sigma-1}(\Omega) \end{array},$$

we have to prove the following fundamental result.

**Proposition 11.** Let  $f \in D^\sigma(\Omega)$  and  $(\phi_\varepsilon)_\varepsilon$  be a net of mollifiers, then

$$\left(f - (f * \phi_\varepsilon)_{/\Omega}\right)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$$

*Proof.* Let  $f \in D^\sigma(\Omega)$ , then there exists a constant  $c > 0$ , such that

$$|\partial^\alpha f(x)| \leq c^{|\alpha|+1} \alpha!^\sigma, \forall \alpha \in \mathbb{Z}_+^m, \forall x \in \Omega$$

Let  $\alpha \in \mathbb{Z}_+^m$ , the Taylor's formula and the properties of  $\phi_\varepsilon$  give

$$\partial^\alpha (f * \phi_\varepsilon - f)(x) = \sum_{|\beta|=N} \int \frac{(\varepsilon y)^\beta}{\beta!} \partial^{\alpha+\beta} f(\xi) \phi(y) dy,$$

where  $x \leq \xi \leq x + \varepsilon y$ . Consequently, for  $b > 0$ , we have

$$\begin{aligned} |\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \varepsilon^N \sum_{|\beta|=N} \int \frac{|y|^N}{\beta!} |\partial^{\alpha+\beta} f(\xi)| |\phi(y)| dy \\ &\leq \alpha!^\sigma \varepsilon^N \sum_{|\beta|=N} \beta!^{2\sigma-1} 2^{\sigma|\alpha+\beta|} b^{|\beta|} \int \frac{|\partial^{\alpha+\beta} f(\xi)|}{(\alpha+\beta)!^\sigma} \times \\ &\quad \times \frac{|y|^{|\beta|}}{b^{|\beta|} \beta!^\sigma} |\phi(y)| dy \end{aligned}$$

Let  $k > 0$  and  $T > 0$ , then

$$\begin{aligned}
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \alpha!^\sigma (\varepsilon N^{2\sigma-1})^N (k^{2\sigma-1}T)^{-N} \times \\
&\times \sum_{|\beta|=N} \int 2^{\sigma|\alpha+\beta|} (k^{2\sigma-1}bT)^{|\beta|} \frac{|\partial^{\alpha+\beta} f(\xi)|}{(\alpha+\beta)!^\sigma} \times \\
&\times \frac{|y|^{|\beta|}}{b^{|\beta|}\beta!^\sigma} |\phi(y)| dy \\
&\leq \alpha!^\sigma (\varepsilon N^{2\sigma-1})^N (k^{2\sigma-1}T)^{-N} \times \\
&\times c \|\phi\|_{b,\sigma} (2^\sigma c)^{|\alpha|} \sum_{|\beta|=N} (2^\sigma k^{2\sigma-1}bT)^{|\beta|} c^{|\beta|},
\end{aligned}$$

hence, taking  $2^\sigma k^{2\sigma-1}bTc \leq \frac{1}{2a}$ , with  $a > 1$ , we obtain

$$\begin{aligned}
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \alpha!^\sigma (\varepsilon N^{2\sigma-1})^N (k^{2\sigma-1}T)^{-N} \times \\
&\times c \|\phi\|_{b,\sigma} (2^\sigma c)^{|\alpha|} a^{-N} \sum_{|\beta|=N} \left(\frac{1}{2}\right)^{|\beta|} \\
(21) \quad &\leq \|\phi\|_{b,\sigma} c^{|\alpha|+1} \alpha!^\sigma (\varepsilon N^{2\sigma-1})^N (k^{2\sigma-1}T)^{-N} a^{-N}
\end{aligned}$$

Let  $\varepsilon_0 \in ]0, 1]$  such that  $\varepsilon_0^{\frac{1}{2\sigma-1}} \frac{\ln a}{k} < 1$  and take  $T > 2^{2\sigma-1}$ , then

$$\left(T^{\frac{1}{2\sigma-1}} - 1\right) > 1 > \frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}}, \forall \varepsilon \leq \varepsilon_0,$$

in particular, we have

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}}\right)^{-1} T^{\frac{1}{2\sigma-1}} - \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}}\right)^{-1} > 1$$

Then, there exists  $N = N(\varepsilon) \in \mathbb{Z}^+$ , such that

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}}\right)^{-1} < N < \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}}\right)^{-1} T^{\frac{1}{2\sigma-1}},$$

i.e.

$$(22) \quad 1 \leq \frac{\ln a}{k} \varepsilon^{\frac{1}{2\sigma-1}} N \leq T^{\frac{1}{2\sigma-1}},$$

which gives

$$a^{-N} \leq \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right) \quad \text{and} \quad \frac{\varepsilon N^{2\sigma-1}}{k^{2\sigma-1}T} \leq \left(\frac{1}{\ln a}\right)^{2\sigma-1} < 1,$$

if we choose  $\ln a > 1$ . Finally, from (21), we have

$$(23) \quad |\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

i.e.  $f * \phi_\varepsilon - f \in \mathcal{N}^\sigma(\Omega)$

□

From the proof, see (21), we obtained in fact the following result.

**Corollary 12.** *Let  $f \in D^\sigma(\Omega)$ , then for every compact  $K$  of  $\Omega, \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \alpha \in \mathbb{Z}_+^m, \forall \varepsilon \leq \varepsilon_0$ ,*

$$(24) \quad \sup_{x \in K} |\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq c^{|\alpha|+1} \alpha!^\sigma \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

### 5. SHEAF PROPERTIES OF $\mathcal{G}^\sigma$

Let  $\Omega'$  be an open subset of  $\Omega$  and let  $f = (f_\varepsilon)_\varepsilon + \mathcal{N}^\sigma(\Omega) \in \mathcal{G}^\sigma(\Omega)$ , the restriction of  $f$  to  $\Omega'$ , denoted  $f_{/\Omega'}$ , is defined as

$$(f_{\varepsilon/\Omega'})_\varepsilon + \mathcal{N}^\sigma(\Omega') \in \mathcal{G}^\sigma(\Omega')$$

**Theorem 13.** *The functor  $\Omega \rightarrow \mathcal{G}^\sigma(\Omega)$  is a sheaf of differential algebras on  $\mathbb{R}^n$ .*

*Proof.* Let  $\Omega$  be a non void open of  $\mathbb{R}^n$  and  $(\Omega_\lambda)_{\lambda \in \Lambda}$  be an open covering of  $\Omega$ . we have to show the properties

- S1) If  $f, g \in \mathcal{G}^\sigma(\Omega)$  such that  $f_{/\Omega_\lambda} = g_{/\Omega_\lambda}, \forall \lambda \in \Lambda$ , then  $f = g$
- S2) If for each  $\lambda \in \Lambda$ , we have  $f_\lambda \in \mathcal{G}^\sigma(\Omega_\lambda)$ , such that

$$f_{\lambda/\Omega_\lambda \cap \Omega_\mu} = f_{\mu/\Omega_\lambda \cap \Omega_\mu} \text{ for all } \lambda, \mu \in \Lambda \text{ with } \Omega_\lambda \cap \Omega_\mu \neq \emptyset,$$

then there exists a unique  $f \in \mathcal{G}^\sigma(\Omega)$  with  $f_{/\Omega_\lambda} = f_\lambda, \forall \lambda \in \Lambda$ .

Let show S1, take  $K$  a compact subset of  $\Omega$ , then there exist compact sets  $K_1, K_2, \dots, K_m$  and indices  $\lambda_1, \lambda_2, \dots, \lambda_m \in \Lambda$  such that

$$K \subset \bigcup_{i=1}^m K_i \text{ and } K_i \subset \Omega_{\lambda_i},$$

where  $(f_\varepsilon - g_\varepsilon)_\varepsilon$  satisfies the  $\mathcal{N}^\sigma$ -estimate on each  $K_i$ , then it satisfies the  $\mathcal{N}^\sigma$ -estimate on  $K$  which means  $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ .

To show S2, let  $(\chi_j)_{j=1}^\infty$  be a  $C^\infty$ -partition of unity subordinate to the covering  $(\Omega_\lambda)_{\lambda \in \Lambda}$ . Set

$$f := (f_\varepsilon)_\varepsilon + \mathcal{N}^\sigma(\Omega),$$

where  $f_\varepsilon = \sum_{j=1}^\infty \chi_j f_{\lambda_j \varepsilon}$  and  $(f_{\lambda_j \varepsilon})_\varepsilon$  is a representative of  $f_{\lambda_j}$ . Moreover, we set  $f_{\lambda_j \varepsilon} = 0$  on  $\Omega \setminus \Omega_{\lambda_j}$ , so that  $\chi_j f_{\lambda_j \varepsilon}$  is  $C^\infty$  on all of  $\Omega$ . First Let  $K$  be compact subset of  $\Omega$ , we have  $K_j = K \cap \text{supp} \chi_j$  is a compact subset of  $\Omega_{\lambda_j}$  and  $(f_{\lambda_j \varepsilon})_\varepsilon \in \mathcal{E}_m^\sigma(\Omega_{\lambda_j})$ , then  $(\chi_j f_{\lambda_j \varepsilon})$  satisfies  $\mathcal{E}_m^\sigma$ -estimate on each  $K_j$ , we have  $\chi_j(x) \equiv 0$  on  $K$  except for finite number of  $j$ , i.e.  $\exists N > 0$ , such that

$$\sum_{j=1}^\infty \chi_j f_{\lambda_j \varepsilon}(x) = \sum_{j=1}^N \chi_j f_{\lambda_j \varepsilon}(x), \forall x \in K$$

So  $(\sum \chi_j f_{\lambda_j \varepsilon})$  satisfies  $\mathcal{E}_m^\sigma$ -estimate on  $K$ , which means  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega)$ . It remains to show that  $f_{/\Omega_\lambda} = f_\lambda, \forall \lambda \in \Lambda$ . Let  $K$  be a compact subset of  $\Omega_\lambda$ , choose  $N > 0$  in such a way that

$\sum_{j=1}^N \chi_j(x) \equiv 1$  on a neighborhood  $\Omega'$  of  $K$  with  $\overline{\Omega'}$  is compact of  $\Omega_\lambda$ . For  $x \in K$ ,

$$f_\varepsilon(x) - f_{\lambda \varepsilon}(x) = \sum_{j=1}^N \chi_j(x) (f_{\lambda_j \varepsilon}(x) - f_{\lambda \varepsilon}(x))$$

Since  $(f_{\lambda_j \varepsilon} - f_{\lambda \varepsilon}) \in \mathcal{N}^\sigma(\Omega_{\lambda_j} \cap \Omega_\lambda)$  and  $K_j = K \cap \text{supp} \chi_j$  is a compact subset of  $\Omega \cap \Omega_{\lambda_j}$ , then  $\left( \sum_{j=1}^N \chi_j (f_{\lambda_j \varepsilon} - f_{\lambda \varepsilon}) \right)$  satisfies the  $\mathcal{N}^\sigma$ -estimate on  $K$ . The uniqueness of such  $f \in \mathcal{G}^\sigma(\Omega)$  follows from S1.  $\square$

Now it is legitimate to introduce the support of  $f \in \mathcal{G}^\sigma(\Omega)$  as in the classical case.

**Definition 7.** *The support of  $f \in \mathcal{G}^\sigma(\Omega)$ , denoted  $\text{supp}_g^\sigma f$ , is the complement of the largest open set  $U$  such that  $f|_U = 0$ .*

As in [7], we construct the embedding of  $D'_{3\sigma-1}(\Omega)$  into  $\mathcal{G}^\sigma(\Omega)$  using the sheaf properties of  $\mathcal{G}^\sigma$ . First, choose some covering  $(\Omega_\lambda)_{\lambda \in \Lambda}$  of  $\Omega$  such that each  $\overline{\Omega_\lambda}$  is a compact subset of  $\Omega$ . Let  $(\psi_\lambda)_{\lambda \in \Lambda}$  be a family of elements of  $D^\sigma(\Omega) \subset D^{3\sigma-1}(\Omega)$  with  $\psi_\lambda \equiv 1$  in some neighborhood of  $\overline{\Omega_\lambda}$ . For each  $\lambda$  we define

$$\begin{aligned} J_\lambda : D'_{3\sigma-1}(\Omega) &\rightarrow \mathcal{G}^\sigma(\Omega_\lambda) \\ T &\rightarrow [T]_\lambda = cl \left( (\psi_\lambda T * \phi_\varepsilon)_{/\Omega_\lambda} \right)_\varepsilon \end{aligned}$$

One can easily show that  $\left( (\psi_\lambda T * \phi_\varepsilon)_{/\Omega_\lambda} \right)_\varepsilon \in \mathcal{E}_m^\sigma(\Omega_\lambda)$ , see the proof of theorem 10, and that the family  $(J_\lambda(T))_{\lambda \in \Lambda}$  is coherent, i.e.

$$J_\lambda(T)_{/\Omega_\lambda \cap \Omega_\mu} = J_\mu(T)_{/\Omega_\lambda \cap \Omega_\mu}, \forall \lambda, \mu \in \Lambda,$$

Then if  $(\chi_j)_{j=1}^\infty$  is a smooth partition of unity subordinate to  $(\Omega_\lambda)_{\lambda \in \Lambda}$ , the precedent theorem allows the embedding

$$(25) \quad \begin{aligned} J_\sigma : D'_{3\sigma-1}(\Omega) &\rightarrow \mathcal{G}^\sigma(\Omega) \\ T &\rightarrow [T] = cl \left( \sum_{j=1}^\infty \chi_j (\psi_{\lambda_j} T * \phi_\varepsilon) \right) \end{aligned}$$

We can also embed canonically  $D'_{3\sigma-1}(\Omega)$  into  $\mathcal{G}^\sigma(\Omega)$ , see [6] for the case  $D'(\Omega)$  and  $\mathcal{G}(\Omega)$ . Indeed, let  $\varphi \in D^\sigma(B(0, 2))$ ,  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $B(0, 1)$  and take  $\phi \in S^{(\sigma)}$ , define the function  $\rho_\varepsilon$  by

$$(26) \quad \rho_\varepsilon(x) = \left( \frac{1}{\varepsilon} \right)^m \phi\left(\frac{x}{\varepsilon}\right) \varphi(x |\ln \varepsilon|)$$

It is easy to prove that,  $\exists c > 0$ , such that  $\forall \alpha \in \mathbb{Z}_+^m$ ,

$$\sup_{x \in \mathbb{R}^m} |\partial^\alpha \rho_\varepsilon(x)| \leq c^{|\alpha|+1} \alpha! \varepsilon^{-m-|\alpha|}$$

Define the injective map

$$(27) \quad \begin{aligned} J : D'_{3\sigma-1}(\Omega) &\rightarrow \mathcal{G}^\sigma(\Omega) \\ T &\rightarrow [T] = cl((T * \rho_\varepsilon)_\varepsilon) \end{aligned}$$

**Proposition 14.** *The map  $J$  coincides on  $E'_{3\sigma-1}(\Omega)$  with  $J_0$ .*

*Proof.* We have to show that for  $T \in E'_{3\sigma-1}(\Omega)$ , the net  $(T * (\rho_\varepsilon - \phi_\varepsilon))_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ . For  $x |\ln \varepsilon| < 1$ , we have  $\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x) = 0$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ . For  $x |\ln \varepsilon| \geq 1$ , we have

$$\begin{aligned} \partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x) &= \varepsilon^{-m-|\alpha|} \partial^\alpha \phi\left(\frac{x}{\varepsilon}\right) (\varphi(x |\ln \varepsilon|) - 1) + \\ &\quad + \varepsilon^{-m} \sum_{|\beta|=1}^{\alpha} \varepsilon^{-|\beta|} |\ln \varepsilon|^{|\alpha-\beta|} \partial^\beta \phi\left(\frac{x}{\varepsilon}\right) \partial^{\alpha-\beta} \varphi(x |\ln \varepsilon|) \end{aligned}$$

Then,  $\exists c > 0, \forall b > 0, \forall \gamma \in \mathbb{Z}_+^m$ ,

$$\begin{aligned} |\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x)| &\leq b^{|\gamma|+|\alpha|+1} \varepsilon^{-m-|\alpha|} \alpha!^\sigma \gamma!^\sigma \left(\frac{\varepsilon}{|x|}\right)^{|\gamma|} + \varepsilon^{-m} \sum_{|\beta|=1}^{\alpha} c^{|\alpha-\beta|+1} b^{|\gamma|+|\beta|+1} \times \\ &\quad \times \varepsilon^{-|\beta|} |\ln \varepsilon|^{|\alpha-\beta|} \beta!^\sigma (\alpha - \beta)!^\sigma \gamma!^\sigma \left(\frac{\varepsilon}{|x|}\right)^{|\gamma|} \\ &\leq b^{|\gamma|+|\alpha|+1} \varepsilon^{-m-|\alpha|} \alpha!^\sigma \gamma!^\sigma (\varepsilon |\ln \varepsilon|)^{|\gamma|} + \varepsilon^{-m} \sum_{|\beta|=1}^{\alpha} b^{|\gamma|+|\beta|+1} c^{|\alpha-\beta|+1} \times \\ &\quad \times \varepsilon^{-|\beta|} |\ln \varepsilon|^{|\alpha-\beta|} \beta!^\sigma (\alpha - \beta)!^\sigma \gamma!^\sigma (\varepsilon |\ln \varepsilon|)^{|\gamma|} \end{aligned}$$

So,  $\exists c > 0, \forall b > 0, \forall \gamma \in \mathbb{Z}_+^m$ ,

$$(28) \quad |\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x)| \leq b^{2|\gamma|+|\alpha|+1} c^{2|\gamma|+|\alpha|+1} \varepsilon^{2|\gamma|-|\alpha|-m} |\ln \varepsilon|^{2|\gamma|} \alpha!^\sigma (2\gamma)!^\sigma$$

As  $\varepsilon \in ]0, 1]$ , we have  $|\ln \varepsilon|^2 \leq \varepsilon^{-1}$ , then  $\forall \gamma \in \mathbb{Z}_+^m$ ,

$$|\ln \varepsilon|^{2|\gamma|} \leq \varepsilon^{-|\gamma|},$$

then (28) gives  $\exists c > 0, \forall b > 0, \forall \gamma \in \mathbb{Z}_+^m$ ,

$$|\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x)| \leq b^{2|\gamma|+|\alpha|+1} c^{2|\gamma|+|\alpha|+1} \alpha!^\sigma (2\gamma)!^\sigma \varepsilon^{|\gamma|-|\alpha|-m}$$

Since

$$(2\gamma)!^\sigma \leq 2^{\sigma|2\gamma|} \gamma!^{2\sigma},$$

then for  $N = |\gamma|$ , we obtain

$$\begin{aligned} |\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x)| &\leq b^{2N+|\alpha|+1} c^{2N+|\alpha|+1} 2^{2N\sigma} \alpha!^\sigma N!^{2\sigma} \varepsilon^{N-|\alpha|-m} \\ &\leq 2^{-N} (2^{2\sigma-1} b^2 c^2)^N N!^{2\sigma} \varepsilon^N \times (bc)^{|\alpha|+1} \alpha!^\sigma \varepsilon^{-|\alpha|-m} \\ &\leq c' h^{|\alpha|+1} \alpha!^\sigma \varepsilon^{-|\alpha|-m} \exp\left(-\left(2^{2\sigma-1} b^2 c^2\right)^{-\frac{1}{2\sigma}} \varepsilon^{-\frac{1}{2\sigma}}\right), \end{aligned}$$

where  $c' = cb \sum_{n \geq 0} 2^{-N}$  and  $h = cb$ , then for any  $k$  take  $b > 0$  and  $\varepsilon$  sufficiently small such that

$$\varepsilon^{-|\alpha|-m} \exp\left(-\left(2^{2\sigma-1} b^2 c^2\right)^{-\frac{1}{2\sigma}} \varepsilon^{-\frac{1}{2\sigma}}\right) \leq \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Then we have  $\exists h > 0, \forall k > 0, \exists \varepsilon_0 \in ]0, 1]$  such that  $\forall \varepsilon \leq \varepsilon_0$ ,

$$(29) \quad |\partial^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x)| \leq h^{|\alpha|+1} \alpha!^\sigma \exp\left(-k \varepsilon^{-\frac{1}{2\sigma}}\right) \leq h^{|\alpha|+1} \alpha!^\sigma \exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

As  $E'_{3\sigma-1}(\Omega) \subset E'_\sigma(\Omega)$ , then  $\exists L$  a compact subset of  $\Omega$  such that  $\forall h > 0, \exists c > 0$ , and

$$|T(y), (\rho_\varepsilon - \phi_\varepsilon)(x - y)| \leq c \sup_{\alpha \in \mathbb{Z}_+^m, y \in L} \frac{h^{|\alpha|}}{\alpha!^\sigma} |\partial_y^\alpha (\rho_\varepsilon - \phi_\varepsilon)(x - y)|$$

So for  $x \in K, y \in L$  and by (29), we obtain

$$|(T * (\rho_\varepsilon - \phi_\varepsilon))(x)| \leq c \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

which proves that  $(T * (\rho_\varepsilon - \phi_\varepsilon))_\varepsilon \in \mathcal{N}^\sigma(\Omega)$ .  $\square$

The sheaf properties of  $\mathcal{G}^\sigma$  and the proof of proposition 14 show that the embedding  $J_\sigma$  coincides with the embedding  $J$ . Summing up, we have the following commutative diagram

$$\begin{array}{ccc} E^\sigma(\Omega) & \rightarrow & \mathcal{G}^\sigma(\Omega) \\ \downarrow & \nearrow & \\ D'_{3\sigma-1}(\Omega) & & \end{array}$$

**Definition 8.** *The space of elements of  $\mathcal{G}^\sigma(\Omega)$  with compact support is denoted  $\mathcal{G}_C^\sigma(\Omega)$ .*

As in the case of Colombeau generalized functions, it is not difficult to prove the following result.

**Proposition 15.** *The space  $\mathcal{G}_C^\sigma(\Omega)$  is the space of elements  $f$  of  $\mathcal{G}^\sigma(\Omega)$  satisfying : there exist a representative  $(f_\varepsilon)_{\varepsilon \in ]0,1]}$  and a compact subset  $K$  of  $\Omega$  such that  $\text{supp} f_\varepsilon \subset K, \forall \varepsilon \in ]0,1]$ .*

## 6. EQUALITIES IN $\mathcal{G}^\sigma(\Omega)$

In  $\mathcal{G}^\sigma(\Omega)$ , we have the strong equality, denoted  $=$ , between two elements  $f = [(f_\varepsilon)_\varepsilon]$  and  $g = [(g_\varepsilon)_\varepsilon]$ , which means that

$$(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}^\sigma(\Omega)$$

One can easily check that if  $K$  is a compact of  $\Omega$  and  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}^\sigma(\Omega)$ , then  $(\int_K f_\varepsilon(x) dx)_\varepsilon$  defines an element of  $\mathcal{C}^\sigma$ .

We define the equality in the sense of ultradistributions, denoted  $\stackrel{t}{\sim}$ , where  $t \in [\sigma, 3\sigma - 1]$ , by

$$f \stackrel{t}{\sim} g \iff \left( \int (f_\varepsilon(x) - g_\varepsilon(x)) \varphi(x) dx \right)_\varepsilon \in \mathcal{N}_0^t, \forall \varphi \in D^t(\Omega),$$

and we say that  $f$  equals  $g$  in the sense of ultradistributions.

We say that  $f = [(f_\varepsilon)_\varepsilon]$  is associated to  $g = [(g_\varepsilon)_\varepsilon]$ , denoted  $f \approx g$ , if

$$\lim_{\varepsilon \rightarrow 0} \int (f_\varepsilon - g_\varepsilon)(x) \psi(x) dx = 0, \forall \psi \in D^{3\sigma-1}(\Omega)$$

In particular, we say that  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}^\sigma(\Omega)$  is associated to the Gevrey ultradistribution  $T \in E'_{3\sigma-1}(\Omega)$ , denoted  $f \approx T$ , if

$$\lim_{\varepsilon \rightarrow 0} \int f_\varepsilon(x) \psi(x) dx = \langle T, \psi \rangle, \forall \psi \in D^{3\sigma-1}(\Omega)$$

The main relationship between these inequalities is giving by the following results.

**Proposition 16.** Let  $f, g \in \mathcal{G}^\sigma(\Omega)$ ,  $T \in E'_{3\sigma-1}(\Omega)$ , and  $t \in [\sigma, 3\sigma - 1]$ , then

- 1)  $f = g \implies f \stackrel{t}{\sim} g \implies f \stackrel{\sigma}{\sim} g \implies f \approx g$
- 2)  $T \approx 0$  in  $\mathcal{G}^\sigma(\Omega) \implies T = 0$  in  $E'_{3\sigma-1}(\Omega)$

*Proof.* Easy. □

## 7. REGULAR GENERALIZED GEVREY ULTRADISTRIBUTIONS

To develop a local and a microlocal analysis with respect to a "good space of regular elements" one needs first to define these regular elements, the notion of singular support and its microlocalization with respect to the class of regular elements.

**Definition 9.** The space of regular elements, denoted  $\mathcal{E}_m^{\sigma,\infty}(\Omega)$ , is the space of  $(f_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^{[0,1]}$  satisfying, for every compact  $K$  of  $\Omega$ ,  $\exists k > 0$ ,  $\exists c > 0$ ,  $\exists \varepsilon_0 \in ]0, 1]$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\forall \varepsilon \leq \varepsilon_0$ ,

$$\sup_{x \in K} |\partial^\alpha f_\varepsilon(x)| \leq c^{|\alpha|+1} \alpha!^\sigma \exp\left(k\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

**Proposition 17.** 1) The space  $\mathcal{E}_m^{\sigma,\infty}(\Omega)$  is an algebra stable by the action of  $\sigma$ -ultradifferential operators.

2) The space  $\mathcal{N}^{\sigma,\infty}(\Omega) := \mathcal{N}^\sigma(\Omega) \cap \mathcal{E}_m^{\sigma,\infty}(\Omega)$  is an ideal of  $\mathcal{E}_m^{\sigma,\infty}(\Omega)$ .

*Proof.* 1) Let  $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\sigma,\infty}(\Omega)$  and  $K$  be a compact of  $\Omega$ , then  $\exists k_1 > 0$ ,  $\exists c_1 > 0$ ,  $\exists \varepsilon_1 \in ]0, 1]$  such that  $\forall x \in K$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\forall \varepsilon \leq \varepsilon_1$ ,

$$|\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} \alpha!^\sigma \exp\left(k_1\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

We have also  $\exists k_2 > 0$ ,  $\exists c_2 > 0$ ,  $\exists \varepsilon_2 \in ]0, 1]$  such that  $\forall x \in K$ ,  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\forall \varepsilon \leq \varepsilon_2$ ,

$$|\partial^\alpha g_\varepsilon(x)| \leq c_2^{|\alpha|+1} \alpha!^\sigma \exp\left(k_2\varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Let  $\alpha \in \mathbb{Z}_+^m$ , then

$$\frac{1}{\alpha!^\sigma} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{1}{(\alpha-\beta)!^\sigma} |\partial^{\alpha-\beta} f_\varepsilon(x)| \frac{1}{\beta!^\sigma} |\partial^\beta g_\varepsilon(x)|$$

Let  $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$  and  $k = k_1 + k_2$ , then we have  $\forall \alpha \in \mathbb{Z}_+^m$ ,  $\forall x \in K$ ,

$$\begin{aligned} \exp\left(-k\varepsilon^{-\frac{1}{2\sigma-1}}\right) \frac{1}{\alpha!^\sigma} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{\exp\left(-k_1\varepsilon^{-\frac{1}{2\sigma-1}}\right)}{(\alpha-\beta)!^\sigma} |\partial^{\alpha-\beta} f_\varepsilon(x)| \\ &\quad \times \frac{\exp\left(-k_2\varepsilon^{-\frac{1}{2\sigma-1}}\right)}{\beta!^\sigma} |\partial^\beta g_\varepsilon(x)| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1^{|\alpha-\beta|} c_2^{|\beta|} \\ &\leq 2^{|\alpha|} (c_1 + c_2)^{|\alpha|}, \end{aligned}$$

i. e.  $(f_\varepsilon)_\varepsilon (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\sigma,\infty}(\Omega)$ .



Let now  $P(D) = \sum a_\gamma D^\gamma$  be an  $\sigma$ -ultradifferential operator, then  $\forall h > 0, \exists b > 0$ , such that

$$\begin{aligned} \exp\left(-k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \frac{1}{\alpha!^\sigma} |\partial^\alpha (P(D) f_\varepsilon(x))| &\leq \exp\left(-k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \sum_{\gamma \in \mathbb{Z}_+^m} b \frac{h^{|\gamma|}}{\gamma!^\sigma} \frac{1}{\alpha!^\sigma} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\ &\leq b \exp\left(-k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \sum_{\gamma \in \mathbb{Z}_+^m} \frac{2^{\sigma|\alpha+\gamma|} h^{|\gamma|}}{(\alpha+\gamma)!^\sigma} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\ &\leq b \sum_{\gamma \in \mathbb{Z}_+^m} 2^{\sigma|\alpha+\gamma|} h^{|\gamma|} c_1^{|\alpha+\gamma|}, \end{aligned}$$

hence, for  $2^\sigma h c_1 \leq \frac{1}{2}$ , we have

$$\exp\left(-k \varepsilon^{-\frac{1}{2\sigma-1}}\right) \frac{1}{\alpha!^\sigma} |\partial^\alpha (P(D) f_\varepsilon(x))| \leq c' (2^\sigma c_1)^{|\alpha|},$$

which shows that  $(P(D) f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\sigma,\infty}(\Omega)$ .

2) The fact that  $\mathcal{N}^{\sigma,\infty}(\Omega) = \mathcal{N}^\sigma(\Omega) \cap \mathcal{E}_m^{\sigma,\infty}(\Omega) \subset \mathcal{E}_m^\sigma(\Omega)$  and  $\mathcal{N}^\sigma(\Omega)$  is an ideal of  $\mathcal{E}_m^\sigma(\Omega)$ , then  $\mathcal{N}^{\sigma,\infty}(\Omega)$  is an ideal of  $\mathcal{E}_m^{\sigma,\infty}(\Omega)$   $\square$

Now, we define the Gevrey regular elements of  $\mathcal{G}^\sigma(\Omega)$ .

**Definition 10.** The algebra of regular generalized Gevrey ultradistributions of order  $\sigma > 1$ , denoted  $\mathcal{G}^{\sigma,\infty}(\Omega)$ , is the quotient algebra

$$\mathcal{G}^{\sigma,\infty}(\Omega) = \frac{\mathcal{E}_m^{\sigma,\infty}(\Omega)}{\mathcal{N}^{\sigma,\infty}(\Omega)}$$

It is clear that  $E^\sigma(\Omega) \hookrightarrow \mathcal{G}^{\sigma,\infty}(\Omega)$ , and it is easy to show that  $\mathcal{G}^{\sigma,\infty}$  is a subsheaf of  $\mathcal{G}^\sigma$ . This motivates the following definition.

**Definition 11.** We define the  $\mathcal{G}^{\sigma,\infty}$ -singular support of a generalized Gevrey ultradistribution  $f \in \mathcal{G}^\sigma(\Omega)$ , denoted  $\sigma$ -sing $\text{supp}_g(f)$ , as the complement of the largest open set  $\Omega'$  such that  $f \in \mathcal{G}^{\sigma,\infty}(\Omega')$ .

The following result is a Paley-Wiener type characterization of  $\mathcal{G}^{\sigma,\infty}(\Omega)$ .

**Proposition 18.** Let  $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_C^\sigma(\Omega)$ , then  $f$  is regular if and only if  $\exists k_1 > 0, \exists k_2 > 0, \exists c > 0, \exists \varepsilon_1 > 0, \forall \varepsilon \leq \varepsilon_1$ , such that

$$(30) \quad |\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^m,$$

where  $\mathcal{F}(f_\varepsilon)$  denote Fourier transform of  $f_\varepsilon$ .

*Proof.* Suppose that  $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_C^\sigma(\Omega) \cap \mathcal{G}^{\sigma,\infty}(\Omega)$ , then  $\exists k_1 > 0, \exists c_1 > 0, \exists \varepsilon_1 > 0, \forall \alpha \in \mathbb{Z}_+^m, \forall x \in K, \forall \varepsilon \leq \varepsilon_1, \text{supp } f_\varepsilon \subset K$ , such that

$$|\partial^\alpha f_\varepsilon| \leq c_1^{|\alpha|+1} \alpha!^\sigma \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

Consequently we have,  $\forall \alpha \in \mathbb{Z}_+^m$ ,

$$|\xi^\alpha| |\mathcal{F}(f_\varepsilon)(\xi)| \leq \left| \int \exp(-ix\xi) \partial^\alpha f_\varepsilon(x) dx \right|,$$

then,  $\exists c > 0, \forall \varepsilon \leq \varepsilon_1$ ,

$$|\xi|^{|\alpha|} |\mathcal{F}(f_\varepsilon)(\xi)| \leq c^{|\alpha|+1} \alpha!^\sigma \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right)$$

For  $\alpha \in \mathbb{Z}_+^m, \exists N \in \mathbb{Z}_+$  such that

$$\frac{N}{\sigma} \leq |\alpha| < \frac{N}{\sigma} + 1,$$

so

$$\begin{aligned} |\xi|^{\frac{N}{\sigma}} |\mathcal{F}(f_\varepsilon)(\xi)| &\leq c^{|\alpha|+1} |\alpha|^{|\alpha|\sigma} \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \\ &\leq c^{N+1} N^N \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \end{aligned}$$

Hence  $\exists c > 0, \forall N \in \mathbb{Z}^+$ ,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c^{N+1} |\xi|^{-\frac{N}{\sigma}} N! \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

which gives

$$|\mathcal{F}(f_\varepsilon)(\xi)| \exp\left(\frac{1}{2c} |\xi|^{\frac{1}{\sigma}}\right) \leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \sum 2^{-N},$$

or

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c' \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - \frac{1}{2c} |\xi|^{\frac{1}{\sigma}}\right),$$

i.e. we have (30).

Suppose now that (30) is valid, then  $\forall \varepsilon \leq \varepsilon_0$ ,

$$|\partial^\alpha f_\varepsilon(x)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \int |\xi^\alpha| \exp\left(-k_2 |\xi|^{\frac{1}{\sigma}}\right) d\xi$$

Due to the inequality  $t^N \leq N! \exp(t), \forall t > 0$ , then  $\exists c_2 = c(k_2)$  such that

$$|\xi^\alpha| \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{\sigma}}\right) \leq c_2^{|\alpha|} \alpha!^\sigma,$$

then

$$\begin{aligned} |\partial^\alpha f_\varepsilon(x)| &\leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) c_2^{|\alpha|} \alpha!^\sigma \int \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{\sigma}}\right) d\xi \\ &\leq c^{|\alpha|+1} \alpha!^\sigma \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right), \end{aligned}$$

where  $c = \max\left(c_1 \int \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{\sigma}}\right) d\xi, c_2\right)$ , i.e.  $f \in \mathcal{G}^{\sigma,\infty}(\Omega)$  □

**Remark 5.** It is easy to see if  $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_C^\sigma(\Omega)$ , then  $\exists k_1 > 0, \exists c > 0, \exists \varepsilon_0 > 0, \forall k_2 > 0, \forall \varepsilon \leq \varepsilon_0$ ,

$$(31) \quad |\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} + k_2 |\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^m$$

The algebra  $\mathcal{G}^{\sigma,\infty}(\Omega)$  plays the same role as the Oberguggenberger subalgebra of regular elements  $\mathcal{G}^\infty(\Omega)$  in the Colombeau algebra  $\mathcal{G}(\Omega)$ , see [17].

**Theorem 19.** We have

$$\mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega) = E^\sigma(\Omega)$$

*Proof.* Let  $S \in \mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega)$ , for any fixed  $x_0 \in \Omega$  we take  $\psi \in D^{3\sigma-1}(\Omega)$  with  $\psi \equiv 1$  on neighborhood  $U$  of  $x_0$ , then  $T = \psi S \in E'_{3\sigma-1}(\Omega)$ . Let  $\phi_\varepsilon$  be a net of mollifiers with  $\check{\phi} = \phi$  and let  $\chi \in D^\sigma(\Omega)$  such that  $\chi \equiv 1$  on  $K = \text{supp}\psi$ . As  $[T] \in \mathcal{G}^{\sigma,\infty}(\Omega)$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_1 > 0, \forall \varepsilon \leq \varepsilon_1$ ,

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \leq c_1 e^{k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}},$$

then

$$\begin{aligned} |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| &= |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(\chi T)(\xi)| \\ &= |\langle T(x), (\chi(x) e^{-i\xi x}) * \phi_\varepsilon(x) - (\chi(x) e^{-i\xi x}) \rangle| \end{aligned}$$

As  $E'_{3\sigma-1}(\Omega) \subset E'_\sigma(\Omega)$ , then  $\exists L$  a compact subset of  $\Omega$  such that  $\forall h > 0, \exists c > 0$ , and

$$|\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi) - \mathcal{F}(T)(\xi)| \leq c \sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{h^{|\alpha|}}{\alpha!^\sigma} |(\partial_x^\alpha (\chi(x) e^{-i\xi x} * \phi_\varepsilon(x) - \chi(x) e^{-i\xi x}))|$$

We have  $e^{-i\xi} \chi \in D^\sigma(\Omega)$ , from the corollary 12,  $\forall k_3 > 0, \exists c_2 > 0, \exists \eta > 0, \forall \varepsilon \leq \eta$ ,

$$\sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{c_2^{|\alpha|}}{\alpha!^\sigma} |\partial_x^\alpha (\chi(x) e^{-i\xi x} * \phi_\varepsilon(x) - \chi(x) e^{-i\xi x})| \leq c_2 e^{-k_3 \varepsilon^{-\frac{1}{2\sigma-1}}},$$

so there exists  $c' = c'(k_3) > 0$ , such that

$$|\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \leq c' e^{-k_3 \varepsilon^{-\frac{1}{2\sigma-1}}}$$

Let  $\varepsilon \leq \min(\eta, \varepsilon_1)$ , then

$$\begin{aligned} |\mathcal{F}(T)(\xi)| &\leq |\mathcal{F}(T)(\xi) - \mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \phi_\varepsilon))(\xi)| \\ &\leq c' e^{-k_3 \varepsilon^{-\frac{1}{2\sigma-1}}} + c_1 e^{k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}} \end{aligned}$$

Take  $c = \max(c', c_1)$ ,  $\varepsilon = \left( \frac{k_1}{(k_2 - r) |\xi|^{\frac{1}{\sigma}}} \right)^{2\sigma-1}$ ,  $r \in ]0, k_2[$  and  $k_3 = \frac{k_1 r}{k_2 - r}$ , then  $\exists \delta > 0, \exists c > 0$  such that

$$|\mathcal{F}(T)(\xi)| \leq c e^{-\delta |\xi|^{\frac{1}{\sigma}}},$$

which means  $T = \psi S \in E^\sigma(\Omega)$ . As  $\psi \equiv 1$  on the neighborhood  $U$  of  $x_0$ , then  $S \in E^\sigma(U)$ , consequently  $S \in E^\sigma(\Omega)$ , which prove.

$$\mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega) \subset E^\sigma(\Omega)$$

We have  $E^\sigma(\Omega) \subset E^{3\sigma-1}(\Omega) \subset D'_{3\sigma-1}(\Omega)$  and  $E^\sigma(\Omega) \subset \mathcal{G}^{\sigma,\infty}(\Omega)$  then

$$E^\sigma(\Omega) \subset \mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega)$$

Consequently we have

$$\mathcal{G}^{\sigma,\infty}(\Omega) \cap D'_{3\sigma-1}(\Omega) = E^\sigma(\Omega)$$

□

## 8. GENERALIZED GEVREY WAVE FRONT

The aim of this section is to introduce the generalized Gevrey wave front of a generalized Gevrey ultradistribution and to give its main properties.

**Definition 12.** We define  $\sum_g^\sigma(f) \subset \mathbb{R}^m \setminus \{0\}$ ,  $f \in \mathcal{G}_C^\sigma(\Omega)$ , as the complement of the set of points having a conic neighborhood  $\Gamma$  such that  $\exists k_1 > 0, \exists k_2 > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1], \forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$ ,

$$(32) \quad |\mathcal{F}(f_\varepsilon)(\xi)| \leq c \exp \left( k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}} \right)$$

The following essential properties of  $\sum_g^\sigma(f)$  are sufficient to define later the generalized Gevrey wave front of generalized Gevrey ultradistribution.

**Proposition 20.** For every  $f \in \mathcal{G}_C^\sigma(\Omega)$ , we have

1. The set  $\sum_g^\sigma(f)$  is a closed cone.
2.  $\sum_g^\sigma(f) = \emptyset \iff f \in \mathcal{G}^{\sigma, \infty}(\Omega)$ .
3.  $\sum_g^\sigma(\psi f) \subset \sum_g^\sigma(f), \forall \psi \in E^\sigma(\Omega)$ .

*Proof.* One can easily, from definition and proposition 18, prove the assertions 1 and 2.

Let suppose that  $\xi_0 \notin \sum_g^\sigma(f)$ , then  $\exists \Gamma$  a conic neighborhood of  $\xi_0$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_1 \in ]0, 1], \forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_1$ ,

$$(33) \quad |\mathcal{F}(f_\varepsilon)(\xi)| \leq c_1 \exp \left( k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}} \right)$$

Let  $\chi \in D^\sigma(\Omega)$ ,  $\chi \equiv 1$  on neighborhood of  $\text{supp} f$ , so  $\chi\psi \in D^\sigma(\Omega)$ , hence  $\exists k_3 > 0, \exists c_2 > 0, \forall \xi \in \mathbb{R}^m$ ,

$$(34) \quad |\mathcal{F}(\chi\psi)(\xi)| \leq c_2 \exp \left( -k_3 |\xi|^{\frac{1}{\sigma}} \right),$$

Let  $\Lambda$  be a conic neighborhood of  $\xi_0$  such that,  $\overline{\Lambda} \subset \Gamma$ , we have, for a fixed  $\xi \in \Lambda$ ,

$$\begin{aligned} \mathcal{F}(\psi f_\varepsilon)(\xi) &= \mathcal{F}(\chi\psi f_\varepsilon)(\xi) \\ &= \int_A \mathcal{F}(f_\varepsilon)(\eta) \mathcal{F}(\chi\psi)(\eta - \xi) d\eta + \int_B \mathcal{F}(f_\varepsilon)(\eta) \mathcal{F}(\chi\psi)(\eta - \xi) d\eta, \end{aligned}$$

where  $A = \left\{ \eta \in \mathbb{R}^m; |\xi - \eta|^{\frac{1}{\sigma}} \leq \delta \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$ ;  $B = \left\{ \eta \in \mathbb{R}^m; |\xi - \eta|^{\frac{1}{\sigma}} > \delta \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$ .

We choose  $\delta$  sufficiently small such that  $A \subset \Gamma$  and  $\frac{|\xi|}{2^\sigma} < |\eta| < 2^\sigma |\xi|, \forall \eta \in A$ . Then for  $\varepsilon \leq \varepsilon_1$ ,

$$\begin{aligned} \left| \int_A \mathcal{F}(f_\varepsilon)(\eta) \mathcal{F}(\chi\psi)(\eta - \xi) d\eta \right| &\leq c_1 c_2 \exp \left( k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - \frac{k_2}{2} |\xi|^{\frac{1}{\sigma}} \right) \times \\ &\quad \times \left| \int_A \exp \left( -k_3 |\eta - \xi|^{\frac{1}{\sigma}} \right) d\eta \right| \\ (35) \quad &\leq c \exp \left( k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - \frac{k_2}{2} |\xi|^{\frac{1}{\sigma}} \right) \end{aligned}$$

As  $f \in \mathcal{G}_C^\sigma(\Omega)$ , from (31),  $\exists c_3 > 0, \exists \mu_1 > 0, \exists \varepsilon_2 \in ]0, 1], \forall \mu_2 > 0, \forall \xi \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_2$ , such that

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c_3 \exp \left( \mu_1 \varepsilon^{-\frac{1}{2\sigma-1}} + \mu_2 |\xi|^{\frac{1}{\sigma}} \right),$$

hence, for  $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$ , we have

$$\begin{aligned} \left| \int_B \mathcal{F}(f_\varepsilon)(\eta) \mathcal{F}(\chi\psi)(\eta - \xi) d\eta \right| &\leq c_2 c_3 \exp\left(\mu_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \int_B \exp\left(\mu_2 |\eta|^\frac{1}{\sigma} - k_3 |\eta - \xi|^\frac{1}{\sigma}\right) d\eta \\ &\leq c \exp\left(\mu_1 \varepsilon^{-\frac{1}{2\sigma-1}}\right) \int_B \exp\left(\mu_2 |\eta|^\frac{1}{\sigma} - k_3 \delta \left(|\xi|^\frac{1}{\sigma} + |\eta|^\frac{1}{\sigma}\right)\right) d\eta, \end{aligned}$$

then, taking  $\mu_2 < k_3 \delta$ , we obtain

$$(36) \quad \left| \int_B \mathcal{F}(f_\varepsilon)(\eta) \mathcal{F}(\chi\psi)(\eta - \xi) d\eta \right| \leq c \exp\left(\mu_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_3 \delta |\xi|^\frac{1}{\sigma}\right)$$

Consequently, (35) and (36) give  $\xi_0 \notin \sum_g^\sigma(\psi f)$ .  $\square$

**Definition 13.** Let  $f \in \mathcal{G}^\sigma(\Omega)$  and  $x_0 \in \Omega$ , the cone of  $\sigma$ -singular directions of  $f$  at  $x_0$ , denoted  $\sum_{g,x_0}^\sigma(f)$ , is

$$(37) \quad \sum_{g,x_0}^\sigma(f) = \bigcap \left\{ \sum_g^\sigma(\phi f) : \phi \in D^\sigma(\Omega) \text{ and } \phi \equiv 1 \text{ on a neighborhood of } x_0 \right\}$$

**Lemma 21.** Let  $f \in \mathcal{G}^\sigma(\Omega)$ , then

$$\sum_{g,x_0}^\sigma(f) = \emptyset \iff x_0 \notin \sigma\text{-singsupp}_g(f)$$

*Proof.* Let  $x_0 \notin \sigma\text{-singsupp}_g(f)$ , i.e.  $\exists U \subset \Omega$  an open neighborhood of  $x_0$  such that  $f \in \mathcal{G}^{\sigma,\infty}(U)$ , let  $\phi \in D^\sigma(U)$  such that  $\phi \equiv 1$  on a neighborhood of  $x_0$ , then  $\phi f \in \mathcal{G}^{\sigma,\infty}(\Omega)$ . Hence, from the proposition 20,  $\sum_g^\sigma(\phi f) = \emptyset$ , i.e.  $\sum_{g,x_0}^\sigma(f) = \emptyset$ .

Suppose now  $\sum_{g,x_0}^\sigma(f) = \emptyset$ , let  $r > 0$  such that  $B(x_0, 2r) \subset \Omega$  and let  $\psi \in D^\sigma(B(x_0, 2r))$  such that  $0 \leq \psi \leq 1$  and  $\psi \equiv 1$  on  $B(x_0, r)$ . Let  $\psi_j(x) = \psi(3^j(x - x_0) + x_0)$  then it is clear that  $\text{supp}(\psi_j) \subset B(x_0, \frac{2r}{3^j}) \subset \Omega$  and  $\psi_j \equiv 1$  on  $B(x_0, \frac{r}{3^j})$ , we have  $\forall \phi \in D^\sigma(\Omega)$  with  $\phi \equiv 1$  on a neighborhood  $U$  of  $x_0$ ,  $\exists j \in \mathbb{Z}^+$  such that  $\text{supp}(\psi_j) \subset U$ , then  $\psi_j f_\varepsilon = \psi_j \phi f_\varepsilon$  and from proposition 20, we have

$$\sum_g^\sigma(\psi_j f) \subset \sum_g^\sigma(\phi f),$$

which gives

$$(38) \quad \bigcap_{j \in \mathbb{Z}^+} \left( \sum_g^\sigma(\psi_j f) \right) = \emptyset$$

We have  $\psi_j \equiv 1$  on  $\text{supp}(\psi_{j+1})$ , then  $\sum_g^\sigma(\psi_{j+1} f) \subset \sum_g^\sigma(\psi_j f)$ , so from (38), there exists  $n \in \mathbb{Z}^+$  sufficiently large such that  $(\psi_n f) \in \mathcal{G}^{\sigma,\infty}(\Omega)$ , then  $f \in \mathcal{G}^{\sigma,\infty}(B(x_0, \frac{r}{3^n}))$ , which means.  $x_0 \notin \sigma\text{-singsupp}_g(f)$ .  $\square$

Now, we are ready to give the definition of the generalized Gevrey wave front.

**Definition 14.** A point  $(x_0, \xi_0) \notin WF_g^\sigma(f) \subset \Omega \times \mathbb{R}^m \setminus \{0\}$  if  $\xi_0 \notin \sum_{g,x_0}^\sigma(f)$ , i.e. there exists  $\phi \in D^\sigma(\Omega)$ ,  $\phi(x) = 1$  neighborhood of  $x_0$ , and conic neighborhood  $\Gamma$  of  $\xi_0$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c > 0, \exists \varepsilon_0 \in ]0, 1]$ , such that  $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$ ,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^\frac{1}{\sigma}\right)$$

The main properties of the generalized Gevrey wave front  $WF_g^\sigma$  are resumed in the following proposition.

**Proposition 22.** *Let  $f \in \mathcal{G}^\sigma(\Omega)$ , then*

- 1) *The projection of  $WF_g^\sigma(f)$  on  $\Omega$  is the  $\sigma$ -sing $\text{supp}_g(f)$ .*
- 2) *If  $f \in \mathcal{G}_C^\sigma(\Omega)$ , then the projection of  $WF_g^\sigma(f)$  on  $\mathbb{R}^m \setminus \{0\}$  is  $\Sigma_g^\sigma(f)$ .*
- 3)  *$\forall \alpha \in \mathbb{Z}_+^m, WF_g^\sigma(\partial^\alpha f) \subset WF_g^\sigma(f)$ .*
- 4)  *$\forall g \in \mathcal{G}^{\sigma,\infty}(\Omega), WF_g^\sigma(gf) \subset WF_g^\sigma(f)$ .*

*Proof.* 1) and 2) hold from the definition, the proposition 20 and lemma 21. 3) Let  $(x_0, \xi_0) \notin WF_g^\sigma(f)$ , then  $\exists \phi \in D^\sigma(\Omega)$ ,  $\phi \equiv 1$  on a neighborhood  $\bar{U}$  of  $x_0$ , there exist a conic neighborhood  $\Gamma$  of  $\xi_0$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_0 \in ]0, 1]$ , such that  $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$ ,

$$(39) \quad |\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}\right)$$

We have, for  $\psi \in D^\sigma(U)$  such that  $\psi(x_0) = 1$ ,

$$\begin{aligned} |\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}((\partial\psi) f_\varepsilon)(\xi)| \\ &\leq |\xi| |\mathcal{F}(\psi \phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial\psi) \phi f_\varepsilon)(\xi)| \end{aligned}$$

As  $WF_g^\sigma(\psi f) \subset WF_g^\sigma(f)$ , then (39) holds for both  $|\mathcal{F}(\psi \phi f_\varepsilon)(\xi)|$  and  $|\mathcal{F}((\partial\psi) \phi f_\varepsilon)(\xi)|$ . So

$$\begin{aligned} |\xi| |\mathcal{F}(\psi \phi f_\varepsilon)(\xi)| &\leq c |\xi| \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}\right) \\ &\leq c' \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_3 |\xi|^{\frac{1}{\sigma}}\right), \end{aligned}$$

with  $c' > 0, k_3 > 0$  such that  $|\xi| \leq c' \exp(k_2 - k_3) |\xi|^{\frac{1}{\sigma}}$ . Hence (39) holds for  $|\mathcal{F}(\psi \partial f_\varepsilon)(\xi)|$ , which proves  $(x_0, \xi_0) \notin WF_g^\sigma(\partial f)$

4) Let  $(x_0, \xi_0) \notin WF_g^\sigma(f)$ , then  $\exists \phi \in D^\sigma(\Omega)$ ,  $\phi \equiv 1$  on a neighborhood  $U$  of  $x_0$ , there exist a conic neighborhood  $\Gamma$  of  $\xi_0$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_0 \in ]0, 1]$ , such that  $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$ ,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}\right)$$

Let  $\psi \in D^\sigma(\Omega)$  and  $\psi \equiv 1$  on  $\text{supp} \phi$ , then  $\mathcal{F}(\phi g_\varepsilon f_\varepsilon) = \mathcal{F}(\psi g_\varepsilon) * \mathcal{F}(\phi f_\varepsilon)$ . We have  $\psi g \in \mathcal{G}^{\sigma,\infty}(\Omega) \cap \mathcal{G}_C^\sigma(\Omega)$ , then  $\exists c_2 > 0, \exists k_3 > 0, \exists k_4 > 0, \exists \varepsilon_1 > 0, \forall \xi \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_1$ ,

$$|\mathcal{F}(\psi g_\varepsilon)(\xi)| \leq c_2 \exp\left(k_3 \varepsilon^{-\frac{1}{2\sigma-1}} - k_4 |\xi|^{\frac{1}{\sigma}}\right),$$

so

$$\mathcal{F}(\phi g_\varepsilon f_\varepsilon)(\xi) = \int_A \mathcal{F}(\phi f_\varepsilon)(\eta) \mathcal{F}(\psi g_\varepsilon)(\eta - \xi) d\eta + \int_B \mathcal{F}(\phi f_\varepsilon)(\eta) \mathcal{F}(\psi g_\varepsilon)(\eta - \xi) d\eta,$$

where  $A$  and  $B$  are the same as in the proof of proposition 20. By (31), we have  $\exists c > 0, \exists \mu_1 > 0, \forall \mu_2 > 0, \exists \varepsilon_2 > 0, \forall \xi \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_2$ ,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c \exp\left(\mu_1 \varepsilon^{-\frac{1}{2\sigma-1}} + \mu_2 |\xi|^{\frac{1}{\sigma}}\right)$$

The same steps as the proposition 20 finish the proof.  $\square$

**Corollary 23.** *Let  $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$  be a partial differential operator with  $\mathcal{G}^{\sigma,\infty}(\Omega)$  coefficients, then*

$$WF_g^\sigma(P(x, D)f) \subset WF_g^\sigma(f), \forall f \in \mathcal{G}^\sigma(\Omega)$$

**Remark 6.** *The reverse inclusion will give a generalized Gevrey microlocal hypoellipticity of linear partial differential operators with regular Gevrey generalized coefficients. The case of generalized  $\mathcal{G}^\infty$ -microlocal hypoellipticity in Colombeau algebra has been studied recently in [12].*

We need the following lemma to show the relationship between  $WF_g^\sigma(T)$  and  $WF^\sigma(T)$ , when  $T \in D'_{3\sigma-1}(\Omega)$ .

**Lemma 24.** *Let  $\varphi \in D^\sigma(B(0, 2))$ ,  $0 \leq \varphi \leq 1$  and  $\varphi \equiv 1$  on  $B(0, 1)$ , and let  $\phi \in S^{(\sigma)}$ , then  $\exists c > 0, \exists \nu > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \in ]0, \varepsilon_0], \forall \xi \in \mathbb{R}^m$ ,*

$$|\widehat{\rho}_\varepsilon(\xi)| \leq c\varepsilon^{-m} e^{-\nu\varepsilon^{\frac{1}{\sigma}}|\xi|^{\frac{1}{\sigma}}},$$

where  $\rho_\varepsilon(x) = \left(\frac{1}{\varepsilon}\right)^m \phi\left(\frac{x}{\varepsilon}\right) \varphi(x|\ln \varepsilon|)$ , and  $\widehat{\rho}$  denotes the Fourier transform of  $\rho$ .

*Proof.* We have, for  $\varepsilon$  sufficiently small,

$$\varepsilon^m \leq |\ln \varepsilon|^{-m} \leq 1$$

Let  $\xi \in \mathbb{R}^m$ , then

$$\begin{aligned} \widehat{\rho}_\varepsilon(\xi) = |\ln \varepsilon|^{-m} & \left[ \int_A \widehat{\phi}(\varepsilon(\xi - \eta)) \widehat{\varphi}\left(\frac{\eta}{|\ln \varepsilon|}\right) d\eta + \right. \\ & \left. \int_B \widehat{\phi}(\varepsilon(\xi - \eta)) \widehat{\varphi}\left(\frac{\eta}{|\ln \varepsilon|}\right) d\eta \right], \end{aligned}$$

where  $A = \left\{ \eta; |\xi - \eta|^{\frac{1}{\sigma}} \leq \delta^{\frac{1}{\sigma}} \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$  and  $B = \left\{ \eta; |\xi - \eta|^{\frac{1}{\sigma}} > \delta^{\frac{1}{\sigma}} \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$ . We choose  $\delta$  sufficiently small such that  $\frac{|\xi|}{2^\sigma} < |\eta| < 2^\sigma |\xi|, \forall \eta \in A_\delta$ . Since  $\varphi \in D^\sigma(\Omega), \phi \in S^{(\sigma)}$ , then  $\exists k_1, k_2 > 0, \exists c_1, c_2 > 0, \forall \xi \in \mathbb{R}^m$ ,

$$|\widehat{\varphi}(\xi)| \leq c_1 \exp\left(-k_1 |\xi|^{\frac{1}{\sigma}}\right) \quad \text{and} \quad \left| \widehat{\phi}(\xi) \right| \leq c_2 \exp\left(-k_2 |\xi|^{\frac{1}{\sigma}}\right)$$

So

$$\begin{aligned} I_1 &= |\ln \varepsilon|^{-m} \left| \int_A \widehat{\phi}(\varepsilon(\xi - \eta)) \widehat{\varphi}\left(\frac{\eta}{|\ln \varepsilon|}\right) d\eta \right| \leq c_1 c_2 \exp\left(-k_2 \frac{|\ln \varepsilon|^{-\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}}{2}\right) \times \\ &\quad \times \int \exp\left(-k_1 \varepsilon^{\frac{1}{\sigma}} |\xi - \eta|^{\frac{1}{\sigma}}\right) d\eta \end{aligned}$$

Let  $z = \varepsilon(\eta - \xi)$ , then

$$I_1 \leq c\varepsilon^{-m} \exp\left(-\frac{k_2}{2} |\ln \varepsilon|^{-\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right) \int \exp\left(-k_1 |z|^{\frac{1}{\sigma}}\right) dz \leq c\varepsilon^{-m} \exp\left(-\nu\varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right)$$

For  $I_2$ , we have

$$\begin{aligned}
I_2 &= |\ln \varepsilon|^{-m} \left| \int_B \widehat{\phi}(\varepsilon(\xi - \eta)) \widehat{\varphi}\left(\frac{\eta}{|\ln \varepsilon|}\right) d\eta \right| \\
&\leq c_1 c_2 \int_B \exp\left(-k_1 \varepsilon^{\frac{1}{\sigma}} |\xi - \eta|^{\frac{1}{\sigma}} - k_2 |\ln \varepsilon|^{-\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) d\eta \\
&\leq c_1 c_2 \exp\left(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right) \int_B \exp\left(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}} - k_2 |\ln \varepsilon|^{-\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) d\eta \\
&\leq c_1 c_2 \exp\left(-k_1 \delta \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right) \int_B \exp\left(-k \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) d\eta \\
&\leq c \varepsilon^{-m} \exp\left(-v \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right)
\end{aligned}$$

Consequently,  $\exists c > 0, \exists v > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0$  such that

$$|\widehat{\rho}_\varepsilon(\xi)| \leq c \varepsilon^{-m} \exp\left(-v \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^m$$

□

We have the following important result.

**Theorem 25.** *Let  $T \in D'_{3\sigma-1}(\Omega) \cap \mathcal{G}^\sigma(\Omega)$ , then  $WF_g^\sigma(T) = WF^\sigma(T)$ .*

*Proof.* Let  $S \in E'_{3\sigma-1}(\Omega) \subset E'_\sigma(\Omega)$  and  $\psi \in D^\sigma(\Omega)$ , we have

$$|\mathcal{F}(\psi(S * \phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| = \left| \left\langle S(x), (\psi(x) e^{-i\xi x}) * \check{\phi}_\varepsilon(x) - (\psi(x) e^{-i\xi x}) \right\rangle \right|,$$

then  $\exists L$  a compact of  $\Omega$  such that  $\forall h > 0, \exists c > 0$ ,

$$|\mathcal{F}(\psi(S * \phi_\varepsilon))(\xi) - \mathcal{F}(\psi S)(\xi)| \leq c \sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{h^{|\alpha|}}{\alpha!^\sigma} \left| \left( \partial_x^\alpha (\psi(x) e^{-i\xi x} * \check{\phi}_\varepsilon(x) - \psi(x) e^{-i\xi x}) \right) \right|$$

We have  $e^{-i\xi} \psi \in D^\sigma(\Omega)$ , from corollary 12,  $\exists c_2 > 0, \forall k_0 > 0, \exists \eta > 0, \forall \varepsilon \leq \eta$ ,

$$(40) \quad \sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{c_2^{|\alpha|}}{\alpha!^\sigma} \left| \partial_x^\alpha (\psi(x) e^{-i\xi x} * \check{\phi}_\varepsilon(x) - \psi(x) e^{-i\xi x}) \right| \leq c_2 e^{-k_0 \varepsilon^{-\frac{1}{2\sigma-1}}},$$

so there exist  $c' > 0, \forall k_0 > 0, \exists \eta > 0, \forall \varepsilon \leq \eta$ , such that

$$(41) \quad |\mathcal{F}(\psi S)(\xi) - \mathcal{F}(\psi(S * \phi_\varepsilon))(\xi)| \leq c' e^{-k_0 \varepsilon^{-\frac{1}{2\sigma-1}}}$$

Let  $T \in D'_{3\sigma-1}(\Omega) \cap \mathcal{G}^\sigma(\Omega)$  and  $(x_0, \xi_0) \notin WF_g^\sigma(T)$ , then there exist  $\chi \in D^\sigma(\Omega)$ ,  $\chi(x) = 1$  in a neighborhood of  $x_0$ , and a conic neighborhood  $\Gamma$  of  $\xi_0$ ,  $\exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_0 \in ]0, 1[$ , such that  $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$ ,

$$(42) \quad |\mathcal{F}(\chi(T * \rho_\varepsilon))(\xi)| \leq c_1 e^{k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}}$$

let  $\psi \in D^\sigma(\Omega)$  equals 1 in neighborhood of  $x_0$  such that for sufficiently small  $\varepsilon$  we have  $\chi \equiv 1$  on  $\text{supp} \psi + B\left(0, \frac{2}{|\ln \varepsilon|}\right)$ , and let  $\varphi \in D^\sigma(B(0, 2))$ ,  $0 \leq \varphi \leq 1$  and  $\varphi \equiv 1$  on  $B(0, 1)$ , then there exist  $\varepsilon_0 < 1$ , such that  $\forall \varepsilon < \varepsilon_0$ ,

$$\psi(T * \rho_\varepsilon)(x) = \psi(\chi T * \rho_\varepsilon)(x)$$



where  $\rho_\varepsilon(x) = \frac{1}{\varepsilon^m} \varphi(x |\ln \varepsilon|) \phi\left(\frac{x}{\varepsilon}\right)$ . As  $\chi T \in E'_{3\sigma-1}(\Omega)$ , then, from proposition 14,

$$\psi(T * \rho_\varepsilon)(x) = \psi(\chi T * \rho_\varepsilon)(x) = \psi(\chi T * \phi_\varepsilon)(x)$$

Let  $\varepsilon \leq \min(\eta, \varepsilon_0)$  and  $\xi \in \Gamma$ , we have

$$\begin{aligned} |\mathcal{F}(\psi T)(\xi)| &\leq |\mathcal{F}(\psi T)(\xi) - \mathcal{F}(\psi(T * \rho_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \rho_\varepsilon))(\xi)| \\ &\leq |\mathcal{F}(\psi \chi T)(\xi) - \mathcal{F}(\psi(\chi T * \phi_\varepsilon))(\xi)| + |\mathcal{F}(\chi(T * \rho_\varepsilon))(\xi)| \end{aligned}$$

then by (41) and (42), we obtain

$$|\mathcal{F}(\psi T)(\xi)| \leq c' e^{-k_0 \varepsilon^{-\frac{1}{2\sigma-1}}} + c_1 e^{k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}}$$

Take  $c = \max(c', c_1)$ ,  $\varepsilon = \left(\frac{k_1}{(k_2 - r) |\xi|^{\frac{1}{\sigma}}}\right)^{2\sigma-1}$ ,  $r \in ]0, k_2[$ ,  $k_0 = \frac{k_1 r}{k_2 - r}$ , then  $\exists \delta > 0, \exists c > 0$ , such that

$$|\mathcal{F}(\chi T)(\xi)| \leq c e^{-\delta |\xi|^{\frac{1}{\sigma}}},$$

which proves that  $(x_0, \xi_0) \notin WF^\sigma(T)$ , i.e.  $WF^\sigma(T) \subset WF_g^\sigma(T)$ .

Suppose  $(x_0, \xi_0) \notin WF^\sigma(T)$ , then there exist  $\chi \in D^\sigma(\Omega)$ ,  $\chi(x) = 1$  in a neighborhood of  $x_0$ , a conic neighborhood  $\Gamma$  of  $\xi_0$ ,  $\exists \lambda > 0, \exists c_1 > 0$ , such that  $\forall \xi \in \Gamma$ ,

$$(43) \quad |\mathcal{F}(\chi T)(\xi)| \leq c_1 e^{-\lambda |\xi|^{\frac{1}{\sigma}}}$$

Let also  $\psi \in D^\sigma(\Omega)$  equals 1 in neighborhood of  $x_0$  such that for sufficiently small  $\varepsilon$  we have  $\chi \equiv 1$  on  $\text{supp} \psi + B\left(0, \frac{2}{|\ln \varepsilon|}\right)$ , then there exist  $\varepsilon_0 < 1$ , such that  $\forall \varepsilon < \varepsilon_0$ ,

$$\psi(T * \rho_\varepsilon)(x) = \psi(\chi T * \rho_\varepsilon)(x)$$

We have

$$\mathcal{F}(\psi(T * \rho_\varepsilon))(\xi) = \int \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta$$

Let  $\Lambda$  be a conic neighborhood of  $\xi_0$  such that,  $\overline{\Lambda} \subset \Gamma$ . For a fixed  $\xi \in \Lambda$ , we have

$$\begin{aligned} \mathcal{F}(\psi(\chi T * \rho_\varepsilon))(\xi) &= \int_A \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta + \\ &\quad \int_B \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta, \end{aligned}$$

where  $A = \left\{ \eta; |\xi - \eta|^{\frac{1}{\sigma}} \leq \delta \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$  and  $B = \left\{ \eta; |\xi - \eta|^{\frac{1}{\sigma}} > \delta \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right) \right\}$ . We choose  $\delta$  sufficiently small such that  $A \subset \Gamma$  and  $\frac{|\xi|}{2^\sigma} < |\eta| < 2^\sigma |\xi|$ . Since  $\psi \in D^\sigma(\Omega)$ , then  $\exists \mu > 0, \exists c_2 > 0, \forall \xi \in \mathbb{R}^m$ ,

$$|\mathcal{F}(\psi)(\xi)| \leq c_2 \exp\left(-\mu |\xi|^{\frac{1}{\sigma}}\right),$$

Then  $\exists c > 0, \exists \varepsilon_0 \in ]0, 1[, \forall \varepsilon \leq \varepsilon_0$ ,

$$\begin{aligned} \left| \int_A \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| &\leq c \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{\sigma}}\right) \times \\ &\quad \times \left| \int_A \exp\left(-\mu |\eta - \xi|^{\frac{1}{\sigma}}\right) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| \end{aligned}$$

From lemma 24,  $\exists c_3 > 0, \exists \nu > 0, \exists \varepsilon_0 > 0$ , such that  $\forall \varepsilon \in ]0, \varepsilon_0]$ ,

$$|\mathcal{F}(\rho_\varepsilon)(\xi)| \leq c_3 \varepsilon^{-m} e^{-\nu \varepsilon^{\frac{1}{\sigma}} |\xi|^{\frac{1}{\sigma}}}, \forall \xi \in \mathbb{R}^m,$$

then  $\exists c > 0$ , such that

$$\begin{aligned} \left| \int_A \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| &\leq c \varepsilon^{-m} \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{\sigma}}\right) \times \\ &\times \int_A \exp\left(-\mu |\eta - \xi|^{\frac{1}{\sigma}}\right) \exp\left(-\nu \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) d\eta \end{aligned}$$

We have  $\exists k > 0, \forall \varepsilon \in ]0, \varepsilon_0]$ ,

$$(44) \quad \varepsilon^{-m} \exp\left(-\nu \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) \leq \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}}\right),$$

so

$$(45) \quad \left| \int_A \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| \leq c \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}} - \frac{\lambda}{2} |\xi|^{\frac{1}{\sigma}}\right)$$

As  $\chi T \in E'_{3\sigma-1}(\Omega) \subset E'_\sigma(\Omega)$ , then  $\forall l > 0, \exists c > 0, \forall \xi \in \mathbb{R}^m$ ,

$$|\mathcal{F}(\chi T)(\xi)| \leq c \exp\left(l |\xi|^{\frac{1}{\sigma}}\right),$$

hence, we have

$$\begin{aligned} \left| \int_B \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| &\leq c \int_B \exp\left(l |\eta|^{\frac{1}{\sigma}} - \mu |\eta - \xi|^{\frac{1}{\sigma}}\right) |\mathcal{F}(\rho_\varepsilon)| d\eta \\ &\leq c' \varepsilon^{-m} \exp\left(-\mu \delta |\xi|^{\frac{1}{\sigma}}\right) \times \\ &\times \int_B \exp\left((l - \mu \delta) |\eta|^{\frac{1}{\sigma}} - \nu \varepsilon^{\frac{1}{\sigma}} |\eta|^{\frac{1}{\sigma}}\right) d\eta, \end{aligned}$$

then, taking  $l - \mu \delta = -a < 0$  and using (44), we obtain for a constant  $c > 0$ ,

$$(46) \quad \left| \int_B \mathcal{F}(\psi)(\xi - \eta) \mathcal{F}(\chi T)(\eta) \mathcal{F}(\rho_\varepsilon)(\eta) d\eta \right| \leq c \exp\left(k \varepsilon^{-\frac{1}{2\sigma-1}} - \mu \delta |\xi|^{\frac{1}{\sigma}}\right)$$

Consequently, (45) and (46) give  $\exists c > 0, \exists k_1 > 0, \exists k_2 > 0$ ,

$$(47) \quad |\mathcal{F}(\psi(T * \rho_\varepsilon))(\xi)| \leq c \exp\left(k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi|^{\frac{1}{\sigma}}\right)$$

which gives that  $(x_0, \xi_0) \notin WF_g^\sigma(T)$ , so  $WF_g^\sigma(T) \subset WF^\sigma(T)$  which ends the proof.  $\square$

## 9. GENERALIZED HÖRMANDER'S THEOREM

To extend the generalized Hörmander's result on the wave front set of the product, define  $WF_g^\sigma(f) + WF_g^\sigma(g)$ , where  $f, g \in \mathcal{G}^\sigma(\Omega)$ , as the set

$$\{(x, \xi + \eta); (x, \xi) \in WF_g^\sigma(f), (x, \eta) \in WF_g^\sigma(g)\}$$

We recall the following fundamental lemma, see [11] for the proof.

**Lemma 26.** Let  $\sum_1, \sum_2$  be closed cones in  $\mathbb{R}^m \setminus \{0\}$ , such that  $0 \notin \sum_1 + \sum_2$ , then

- i)  $\overline{\sum_1 + \sum_2}^{\mathbb{R}^m \setminus \{0\}} = (\sum_1 + \sum_2) \cup \sum_1 \cup \sum_2$
- ii) For any open conic neighborhood  $\Gamma$  of  $\sum_1 + \sum_2$  in  $\mathbb{R}^m \setminus \{0\}$ , one can find open conic neighborhoods of  $\Gamma_1, \Gamma_2$  in  $\mathbb{R}^m \setminus \{0\}$  of, respectively,  $\sum_1, \sum_2$ , such that

$$\Gamma_1 + \Gamma_2 \subset \Gamma$$

The principal result of this section is the following theorem.

**Theorem 27.** Let  $f, g \in \mathcal{G}^\sigma(\Omega)$ , such that  $\forall x \in \Omega$ ,

$$(48) \quad (x, 0) \notin WF_g^\sigma(f) + WF_g^\sigma(g),$$

then

$$(49) \quad WF_g^\sigma(fg) \subseteq (WF_g^\sigma(f) + WF_g^\sigma(g)) \cup WF_g^\sigma(f) \cup WF_g^\sigma(g)$$

*Proof.* Let  $(x_0, \xi_0) \notin (WF_g^\sigma(f) + WF_g^\sigma(g)) \cup WF_g^\sigma(f) \cup WF_g^\sigma(g)$ , then  $\exists \phi \in D^\sigma(\Omega)$ ,  $\phi(x_0) = 1$ ,  $\xi_0 \notin \left( \sum_g^\sigma(\phi f) + \sum_g^\sigma(\phi g) \right) \cup \sum_g^\sigma(\phi f) \cup \sum_g^\sigma(\phi g)$ . From (48) we have  $0 \notin \sum_g^\sigma(\phi f) + \sum_g^\sigma(\phi g)$  then by lemma 26 i), we have

$$\xi_0 \notin \left( \sum_g^\sigma(\phi f) + \sum_g^\sigma(\phi g) \right) \cup \sum_g^\sigma(\phi f) \cup \sum_g^\sigma(\phi g) = \overline{\sum_g^\sigma(\phi f) + \sum_g^\sigma(\phi g)}^{\mathbb{R}^m \setminus \{0\}}$$

Let  $\Gamma_0$  be an open conic neighborhood of  $\sum_g^\sigma(\phi f) + \sum_g^\sigma(\phi g)$  in  $\mathbb{R}^m \setminus \{0\}$  such that  $\xi_0 \notin \bar{\Gamma}_0$  then, from lemma 26 ii), there exist open cones  $\Gamma_1$  and  $\Gamma_2$  in  $\mathbb{R}^m \setminus \{0\}$  such that

$$\sum_g^\sigma(\phi f) \subset \Gamma_1, \quad \sum_g^\sigma(\phi g) \subset \Gamma_2 \text{ and } \Gamma_1 + \Gamma_2 \subset \Gamma_0$$

Define  $\Gamma = \mathbb{R}^m \setminus \bar{\Gamma}_0$ , so

$$(50) \quad \Gamma \cap \Gamma_2 = \emptyset \text{ and } (\Gamma - \Gamma_2) \cap \Gamma_1 = \emptyset$$

Let  $\xi \in \Gamma$  and  $\varepsilon \in ]0, 1]$

$$\begin{aligned} \mathcal{F}(\phi f_\varepsilon \phi g_\varepsilon)(\xi) &= (\mathcal{F}(\phi f_\varepsilon) * \mathcal{F}(\phi g_\varepsilon))(\xi) \\ &= \int_{\Gamma_2} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta + \int_{\Gamma_2^c} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_1(\xi) + I_2(\xi) \end{aligned}$$

From (50),  $\exists c_1 > 0, \exists k_1, k_2 > 0, \exists \varepsilon_1 > 0$  such that  $\forall \varepsilon \leq \varepsilon_1, \forall \eta \in \Gamma_2$ ,

$$\mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \leq c_1 \exp \left( k_1 \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 |\xi - \eta|^{\frac{1}{\sigma}} \right),$$

and from (31)  $\exists c_2 > 0, \exists k_3 > 0, \forall k_4 > 0, \exists \varepsilon_2 > 0, \forall \eta \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_2$ ,

$$|\mathcal{F}(\phi g_\varepsilon)(\eta)| \leq c_2 \exp \left( k_2 \varepsilon^{-\frac{1}{2\sigma-1}} + k_4 |\eta|^{\frac{1}{\sigma}} \right)$$

Let  $\gamma > 0$  sufficiently small such that  $|\xi - \eta|^{\frac{1}{\sigma}} \geq \gamma \left( |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} \right), \forall \eta \in \Gamma_2$ . Hence for  $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$ ,

$$|I_1(\xi)| \leq c_1 c_2 \exp \left( (k_1 + k_2) \varepsilon^{-\frac{1}{2\sigma-1}} - k_2 \gamma |\xi|^{\frac{1}{\sigma}} \right) \int \exp \left( -k_2 \gamma |\eta|^{\frac{1}{\sigma}} + k_4 |\eta|^{\frac{1}{\sigma}} \right) d\eta$$

take  $k_4 > k_2\gamma$ , then

$$(51) \quad |I_1(\xi)| \leq c' \exp \left( k_1' \varepsilon^{-\frac{1}{2\sigma-1}} - k_2' |\xi|^{\frac{1}{\sigma}} \right)$$

Let  $r > 0$ ,

$$\begin{aligned} I_2(\xi) &= \int_{\Gamma_2 \cap \{|\eta| \leq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta + \int_{\Gamma_2 \cap \{|\eta| \geq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_{21}(\xi) + I_{22}(\xi) \end{aligned}$$

Choose  $r$  sufficiently small such that  $\left\{ |\eta|^{\frac{1}{\sigma}} \leq r |\xi|^{\frac{1}{\sigma}} \right\} \implies \xi - \eta \notin \Gamma_1$ . Then  $|\xi - \eta|^{\frac{1}{\sigma}} \geq (1-r) |\xi|^{\frac{1}{\sigma}} \geq (1-2r) |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}}$ , consequently  $\exists c_3 > 0, \exists \lambda_1, \lambda_2, \lambda_3 > 0, \exists \varepsilon_3 > 0$  such that  $\forall \varepsilon \leq \varepsilon_3$ ,

$$\begin{aligned} |I_{21}(\xi)| &\leq c_3 \exp \left( \lambda_1 \varepsilon^{-\frac{1}{2\sigma-1}} \right) \int \exp \left( -\lambda_2 |\xi - \eta|^{\frac{1}{\sigma}} - \lambda_3 |\eta|^{\frac{1}{\sigma}} \right) d\eta \\ &\leq c_3 \exp \left( \lambda_1 \varepsilon^{-\frac{1}{2\sigma-1}} - \lambda_2' |\xi|^{\frac{1}{\sigma}} \right) \int \exp \left( -\lambda_3' |\eta|^{\frac{1}{\sigma}} \right) d\eta \\ &\leq c_3' \exp \left( \lambda_1 \varepsilon^{-\frac{1}{2\sigma-1}} - \lambda_2' |\xi|^{\frac{1}{\sigma}} \right) \end{aligned}$$

If  $|\eta|^{\frac{1}{\sigma}} \geq r |\xi|^{\frac{1}{\sigma}}$ , we have  $|\eta|^{\frac{1}{\sigma}} \geq \frac{|\eta|^{\frac{1}{\sigma}} + r |\xi|^{\frac{1}{\sigma}}}{2}$ , and then  $\exists c_4 > 0, \exists \mu_1, \mu_3 > 0, \forall \mu_2 > 0, \exists \varepsilon_4 > 0$  such that  $\forall \varepsilon \leq \varepsilon_4$ ,

$$\begin{aligned} |I_{21}(\xi)| &\leq c_4 \exp \left( \mu_1 \varepsilon^{-\frac{1}{2\sigma-1}} \right) \int \exp \left( \mu_2 |\xi - \eta|^{\frac{1}{\sigma}} - \mu_3 |\eta|^{\frac{1}{\sigma}} \right) d\eta \\ &\leq c_4 \exp \left( \mu_1 \varepsilon^{-\frac{1}{2\sigma-1}} \right) \int \exp \left( \mu_2 |\xi - \eta|^{\frac{1}{\sigma}} - \mu_3' |\eta|^{\frac{1}{\sigma}} - \mu_3' |\xi|^{\frac{1}{\sigma}} \right) d\eta, \end{aligned}$$

if take  $\mu_2 < \frac{\mu_3'}{2} \left( 1 + \frac{1}{r} \right)$ , we obtain

$$|I_{21}(\xi)| \leq c_4' \exp \left( k_3' \varepsilon^{-\frac{1}{2\sigma-1}} - \mu_3' |\xi|^{\frac{1}{\sigma}} \right),$$

which finishes the proof.  $\square$

## REFERENCES

- [1] A. B. Antonevich, Ya. V. Radyno. On a general method of constructing algebras of new generalized functions. Soviet. Math. Dokl., vol. 43:3, (1991), 680-684.
- [2] K. Benmeriem, C. Bouzar. Colombeau generalized functions and solvability of differential operators. Z. Anal. Anw. 25:4 (2006), 467-477.
- [3] K. Benmeriem, C. Bouzar. Ultraregular generalized functions. Oran-Essenia University, Preprint 2006.
- [4] J. F. Colombeau. Elementary introduction to new generalized functions. North Holland, 1984.
- [5] J. F. Colombeau. Multiplication of Distributions: a tool in mathematics numerical engineering and theoretical physics. Lecture Notes in Math. 1532, Springer, 1992.
- [6] A. Delcroix. Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions. Novi Sad J. Math., vol. 35, n°2, (2005), 27-40.
- [7] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. Geometric theory of generalized functions with applications to general relativity, Kluwer Publishing, 2001.
- [8] I. M. Guelfand, G. E. Shilov. Generalized functions, vol. 2, Academic Press, 1967.
- [9] T. Gramchev. Nonlinear maps in space of distributions, Math. Z., 209, (1992), 101-114.
- [10] L. Hörmander. Distribution theory and Fourier analysis, Springer, 1983.

- [11] G. Hörmann, M. Kunzinger. Microlocal properties of basic operations in Colombeau algebras. J. Math. Anal. Appl., 261, (2001), 254-270.
- [12] G. Hörmann, M. Oberguggenberger, S. Pilipović. Microlocal hypoellipticity of linear differential operators with generalized functions as coefficients, Trans. Amer. Math. Soc., vol. 358, n° 8, (2005), 3363-3383.
- [13] H. Komatsu. Ultradistributions I, J. Fac. Sci. Univ. Tokyo, Sect. IA, 20, (1973), 25-105.
- [14] J. L. Lions, E. Magenes. Non-homogeneous boundary value problems and applications, vol.3, Springer, 1973.
- [15] J. A. Marti.  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -Sheaf structures and applications. In M. Grosser, G. Hörmann, M. Kunzinger and M. Oberguggenberger (Editors) Nonlinear Theory of Generalized Functions. pp 175-186. Chapman and Hall. 1999
- [16] M. Nedeljkov, S. Pilipovic, D. Scarpalézos. The linear theory of Colombeau generalized functions, Longman Scientific & Technica, 1998.
- [17] M. Oberguggenberger. Multiplication of distributions and applications to partial differential equations, Longman Scientific & Technical, 1992.
- [18] S. Pilipovic, D. Scarpalézos. Colombeau generalized ultradistributions. Math. Proc. Camb. Phil. Soc., 130, (2001), 541-553
- [19] L. Rodino. Linear partial differential operators in Gevrey spaces. World Scientific. 1993.
- [20] L. Schwartz. Sur l'impossibilité de la multiplication des distributions. C. R. Acad. Sci., 239, (1954), 847-848.

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